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String instantons, fluxes and moduli stabilization

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Abstract

We analyze a class of dual pairs of heterotic and type I models based on freely-acting $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds in four dimensions. Using the adiabatic argument, it is possible to calculate non-perturbative contributions to the gauge coupling threshold corrections on the type I side by exploiting perturbative calculations on the heterotic side, without the drawbacks due to twisted moduli. The instanton effects can then be combined with closed-string fluxes to stabilize most of the moduli fields of the internal manifold, and also the dilaton, in a racetrack realization of the type I model.

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1. Introduction and conclusions

In recent years new ways to compute non-perturbative effects in string theory were developed, based on Euclidean p-branes (Ep-branes) wrapping various cycles of the internal manifold of string compactifications [1–7]. Some of the instanton effects have an interpretation in terms of gauge theory instantons, whereas others are stringy instanton effects whose gauge theory counter-

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part is still under investigation. (For recent reviews on instanton effects in field and string theory, see, e.g., [8].) Whereas the former effects are responsible for the generation of non-perturbative superpotentials via gauge theory strong IR dynamics [9] and of moduli potentials satisfying various gauge invariance constraints [10], the latter could be responsible for generating Majorana neutrino masses or the μ-term in MSSM [4,5], as well as for inducing other interesting effects at low energy [7].

The purpose of the present paper is to present a class of examples based on freely-acting $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold models, that adds two new ingredients to the discussion, trying to go deeper into the non-perturbative effects analysis. The first new ingredient is the heterotic-type I duality [11], which exchanges perturbative and non-perturbative regimes. As is well known [12], it is possible to construct freely-acting dual pairs with $\mathcal{N} = 1$ supersymmetry in four dimensions which preserve the S-duality structure. As we show explicitly here, the dual pairs can have a rich non-perturbative dynamics exhibiting both types of effects mentioned above. The heterotic-type I duality allows, for example, to obtain the exact E1 instantonic summations on the type I side for the non-perturbative corrections to the gauge couplings using the computation of perturbative threshold corrections on the heterotic side.[3] Second, non-perturbative effects also play a potentially important role in addressing the moduli field stabilization issue. Closed string fluxes were invoked in recent years in the framework of type IIB and type IIA string compactifications, following the initial proposal of [14] to try to stabilize all moduli fields, including the dilaton. The combination of closed string fluxes and freely-acting orbifold actions has the obvious advantage of avoiding to deal with twisted-sector moduli fields, absent in our construction. We show that, besides the Ramond–Ramond (RR) three-form fluxes, also metric fluxes can be turned on in our freely-acting type I models, requiring new quantization conditions and the twisting of the cohomology of the internal manifold. The low-energy effective description is equivalent to the original one, with the addition of a non-trivial superpotential. Moreover, our string constructions allow naturally racetrack models with dilaton stabilization [15]. We show how they can be combined with closed string fluxes and stringy instanton effects in order to stabilize most of the moduli fields of the internal manifold.

The plan of the paper is as follows. In Section 2 we discuss the geometric framework of the freely acting $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds. In Section 3 we display the explicit type I descendants obtained by quotienting the orbifold with the geometric world-sheet parity operator. Besides some variations of the simplest class with orthogonal gauge groups, we also construct the corresponding heterotic duals in Section 4. In Section 5, we report the calculation of the threshold corrections to the gauge couplings both for the heterotic and for the type I models. The details of the calculations are reported in Appendices A–D. In particular, we verify that the moduli dependence of the non-perturbative corrections on the type I side is in agreement with the conjectured form [16]. In Section 6 we analyze the instanton contributions in the type I framework, that are combined with closed string fluxes in Section 7 in order to attain the stabilization of most of the moduli of the compactification manifold. In particular, in Section 7 we describe an example in which the dilaton can be also stabilized, due to a natural racetrack realization of the type I model in combination with closed metric and RR three-form fluxes.

[3] See [13] for earlier work on instanton effects and heterotic-type I duality.

2. The freely-acting orbifold and its moduli

From the point of view of the target space, we take a T^6 torus ($y^i = y^i + 1$) with vielbein vectors $e^i = e^i{}_\mu \, dy^\mu$ and metric given by

$$ds^2 = \sum_i e^i{}_\mu e^{i\mu}. \tag{2.1}$$

This has to be $SL(6,\mathbb{Z})$ invariant. Therefore, performing a general rotation of the lattice vectors one may write a basis as follows[4]

$$e^6 = R^6\, dy^6, \tag{2.2}$$
$$e^5 = R^5\big(dy^5 + a^5{}_6\, dy^6\big), \tag{2.3}$$
$$e^4 = R^4\big(dy^4 + a^4{}_5\, dy^5 + a^4{}_6\, dy^6\big), \tag{2.4}$$
$$e^3 = R^3\big(dy^3 + a^3{}_4\, dy^4 + a^3{}_5\, dy^5 + a^3{}_6\, dy^6\big), \tag{2.5}$$
$$e^2 = R^2\big(dy^2 + a^2{}_3\, dy^3 + a^2{}_4\, dy^4 + a^2{}_5\, dy^5 + a^2{}_6\, dy^6\big), \tag{2.6}$$
$$e^1 = R^1\big(dy^1 + a^1{}_2\, dy^2 + a^1{}_3\, dy^3 + a^1{}_4\, dy^4 + a^1{}_5\, dy^5 + a^1{}_6\, dy^6\big). \tag{2.7}$$

Modding by the orbifold action will break the $SL(6,\mathbb{Z})$ symmetry to a smaller subgroup. We define the generators $\{g, f, h\}$ of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ freely-acting orbifold as,

$$\big(y^1, y^2, y^3, y^4, y^5, y^6\big) \xrightarrow{g} \big(y^1 + 1/2, y^2, -y^3, -y^4, -y^5 + 1/2, -y^6\big), \tag{2.8}$$
$$\big(y^1, y^2, y^3, y^4, y^5, y^6\big) \xrightarrow{f} \big(-y^1 + 1/2, -y^2, y^3 + 1/2, y^4, -y^5, -y^6\big), \tag{2.9}$$
$$\big(y^1, y^2, y^3, y^4, y^5, y^6\big) \xrightarrow{h} \big(-y^1, -y^2, -y^3 + 1/2, -y^4, y^5 + 1/2, y^6\big). \tag{2.10}$$

Notice that these orbifold operations have no fixed points due to the shifts, hence they act freely (see, e.g., [17]). Moreover, for objects localized in the internal space, as will be the case for the $\mathrm{E}1_i$ instantons to be discussed in Section 6, orbifold operations will generate inevitably instanton images. This has non-trivial consequences on the instanton spectra, as we shall see later on.

In order for the lattice vectors (2.2)–(2.7) to transform covariantly with respect to the orbifold action, it is required that

$$a^4{}_5 = a^4{}_6 = a^3{}_5 = a^3{}_6 = a^2{}_3 = a^2{}_4 = a^2{}_5 = a^2{}_6 = a^1{}_3 = a^1{}_4 = a^1{}_5 = a^1{}_6 = 0. \tag{2.11}$$

A basis of holomorphic vectors can thus be introduced in the form

$$z^1 = e^1 + ie^2 = R_1\big(dy^1 + iU_1\, dy^2\big), \tag{2.12}$$
$$z^2 = e^3 + ie^4 = R_3\big(dy^3 + iU_2\, dy^4\big), \tag{2.13}$$
$$z^3 = e^5 + ie^6 = R_5\big(dy^5 + iU_3\, dy^6\big), \tag{2.14}$$

where we have defined

$$U_1 = \frac{R^2}{R^1} - ia^1{}_2, \qquad U_2 = \frac{R^4}{R^3} - ia^3{}_4, \qquad U_3 = \frac{R^6}{R^5} - ia^5{}_6. \tag{2.15}$$

[4] We use the notation y^i, $i = 1, \ldots, 6$, to denote the internal compact dimensions and x^i, $i = 0, \ldots, 3$, for the non-compact space–time dimensions.

Hence, the moduli space of the untwisted sector matches precisely the one of an ordinary $\mathbb{Z}_2 \times \mathbb{Z}_2$, given by the three complex structure moduli, U_i, together with the three Kähler moduli, T_i, which result from the expansion of the complexified Kähler 2-form in a cohomology basis of even 2-forms,

$$J_c = e^{-\phi} J + i C_2 = T_1\, dy^1 \wedge dy^2 + T_2\, dy^3 \wedge dy^4 + T_3\, dy^5 \wedge dy^6. \tag{2.16}$$

Making use of (2.12)–(2.14), the real parts of the Kähler moduli can be seen to be

$$\operatorname{Re} T_1 = e^{-\phi} R_1 R_2, \qquad \operatorname{Re} T_2 = e^{-\phi} R_3 R_4, \qquad \operatorname{Re} T_3 = e^{-\phi} R_5 R_6. \tag{2.17}$$

The effective theory contains also, as usual, the universal axion-dilaton modulus

$$S = e^{-\phi} \prod_{i=1}^{6} R_i + i c, \tag{2.18}$$

where c is the universal axion. On the other hand, since there are no fixed points in the orbifold action, we expect the twisted sector to be trivial. We shall see in next section, from the exchange of massless modes in the vacuum amplitudes, that this is indeed the case. The internal space of the orbifold is therefore completely smooth and can be interpreted as a Calabi–Yau space with Hodge numbers $(h_{11}, h_{21}) = (3, 3)$. The corresponding type IIB string theory on this orbifold space has the standard left–right worldsheet involution Ω_P as a symmetry, which we use, following [18,19], in order to construct type I freely-acting orbifolds.

3. Type I models: Vacuum energy and spectra

3.1. Type I with orthogonal gauge groups

We briefly summarize here some of the results of [18]. Following the original notation, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold generators of Eqs. (2.8)–(2.10) can be written as

$$g = (P_1, -1, -P_3), \qquad f = (-P_1, P_2, -1), \qquad h = (-1, -P_2, P_3), \tag{3.1}$$

where P_i represents the momentum shift along the real direction y^{2i-1} of the ith torus. We consider the type I models obtained by gauging the type IIB string with Ω_P, the standard worldsheet orientifold involution. The spectrum can be read from the one-loop amplitudes [20]. In particular, the torus partition function is[5]

$$\begin{aligned} T = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^3 |\eta|^4} \frac{1}{4} \biggl[& |\tau_{oo} + \tau_{og} + \tau_{oh} + \tau_{of}|^2 \Lambda_1 \Lambda_2 \Lambda_3 \\ & + |\tau_{oo} + \tau_{og} - \tau_{oh} - \tau_{of}|^2 (-1)^{m_1} \Lambda_1 \left| \frac{4\eta^2}{\vartheta_2^2} \right|^2 \\ & + |\tau_{oo} - \tau_{og} + \tau_{oh} - \tau_{of}|^2 (-1)^{m_3} \Lambda_3 \left| \frac{4\eta^2}{\vartheta_2^2} \right|^2 \end{aligned}$$

[5] There is an overall normalization that is explicitly written in Appendix A. For other conventions concerning orientifolds, see, e.g., the reviews [21].

$$
\begin{aligned}
&+|\tau_{oo}-\tau_{og}-\tau_{oh}+\tau_{of}|^2(-1)^{m_2}\Lambda_2\left|\frac{4\eta^2}{\vartheta_2^2}\right|^2\\
&+|\tau_{go}+\tau_{gg}+\tau_{gh}+\tau_{gf}|^2\Lambda_1^{n_1+\frac{1}{2}}\left|\frac{4\eta^2}{\vartheta_4^2}\right|^2\\
&+|\tau_{go}+\tau_{gg}-\tau_{gh}-\tau_{gf}|^2(-1)^{m_1}\Lambda_1^{n_1+\frac{1}{2}}\left|\frac{4\eta^2}{\vartheta_3^2}\right|^2\\
&+|\tau_{ho}+\tau_{hg}+\tau_{hh}+\tau_{hf}|^2\Lambda_3^{n_3+\frac{1}{2}}\left|\frac{4\eta^2}{\vartheta_4^2}\right|^2\\
&+|\tau_{ho}-\tau_{hg}+\tau_{hh}-\tau_{hf}|^2(-1)^{m_3}\Lambda_3^{n_3+\frac{1}{2}}\left|\frac{4\eta^2}{\vartheta_3^2}\right|^2\\
&+|\tau_{fo}+\tau_{fg}+\tau_{fh}+\tau_{ff}|^2\Lambda_2^{n_2+\frac{1}{2}}\left|\frac{4\eta^2}{\vartheta_4^2}\right|^2\\
&+|\tau_{fo}-\tau_{fg}-\tau_{fh}+\tau_{ff}|^2(-1)^{m_2}\Lambda_2^{n_2+\frac{1}{2}}\left|\frac{4\eta^2}{\vartheta_3^2}\right|^2\Bigg],
\end{aligned}
\tag{3.2}
$$

while the Klein bottle, annulus and Möbius strip amplitudes read in the direct (loop) channel respectively as

$$
\begin{aligned}
K=\int_0^\infty \frac{dt}{t^3\eta^2}\frac{1}{8}(\tau_{oo}+\tau_{og}+\tau_{oh}+\tau_{of})\big\{&P_1P_2P_3+(-1)^{m_1}P_1W_2W_3\\
&+W_1(-1)^{m_2}P_2W_3+W_1W_2(-1)^{m_3}P_3\big\},
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
A=\int_0^\infty \frac{dt}{t^3\eta^2}\frac{1}{8}\bigg\{&I_N^2(\tau_{oo}+\tau_{og}+\tau_{oh}+\tau_{of})P_1P_2P_3\\
&+g_N^2(\tau_{oo}+\tau_{og}-\tau_{oh}-\tau_{of})(-1)^{m_1}P_1\frac{4\eta^2}{\vartheta_2^2}\\
&+h_N^2(\tau_{oo}-\tau_{og}+\tau_{oh}-\tau_{of})(-1)^{m_3}P_3\frac{4\eta^2}{\vartheta_2^2}\\
&+f_N^2(\tau_{oo}-\tau_{og}-\tau_{oh}+\tau_{of})(-1)^{m_2}P_2\frac{4\eta^2}{\vartheta_2^2}\bigg\},
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
M=-\int_0^\infty \frac{dt}{t^3\eta^2}\frac{I_N}{8}\bigg\{&(\hat\tau_{oo}+\hat\tau_{og}+\hat\tau_{oh}+\hat\tau_{of})P_1P_2P_3\\
&+(\hat\tau_{oo}+\hat\tau_{og}-\hat\tau_{oh}-\hat\tau_{of})(-1)^{m_1}P_1\frac{4\hat\eta^2}{\hat\vartheta_2^2}+(\hat\tau_{oo}-\hat\tau_{og}+\hat\tau_{oh}-\hat\tau_{of})(-1)^{m_3}P_3\frac{4\hat\eta^2}{\hat\vartheta_2^2}\\
&+(\hat\tau_{oo}-\hat\tau_{og}-\hat\tau_{oh}+\hat\tau_{of})(-1)^{m_2}P_2\frac{4\hat\eta^2}{\hat\vartheta_2^2}\bigg\}.
\end{aligned}
\tag{3.5}
$$

Some comments on the notation are to be made. In the torus amplitude, $\mathcal{F}$ is the fundamental domain and the Λ_i are the lattice sums for the three compact tori, whereas the shorthand notation $(-1)^{m_i}\Lambda_i^{n_i+1/2}$ indicates a sum with the insertion of $(-1)^{m_i}$ along the momentum in y^{2i-1}, with the corresponding winding number shifted by $1/2$. P_i and W_i in (3.3)–(3.5) are respectively the momentum and winding sums for the three two-dimensional tori. More concretely, using for the geometric moduli the conventions of the previous section, one has[6]

$$P_i \equiv \sum_{m,m'} \exp\left[-\frac{\pi t}{(\operatorname{Re} T_i)(\operatorname{Re} U_i)}|m' - iU_i m|^2\right], \tag{3.6}$$

$$(-1)^{m_i} P_i \equiv \sum_{m,m'} (-1)^m \exp\left[-\frac{\pi t}{(\operatorname{Re} T_i)(\operatorname{Re} U_i)}|m' - iU_i m|^2\right]. \tag{3.7}$$

Moreover, in (3.5) hatted modular functions define a correct basis under the P transformation extracting a suitable overall phase [20]. Indeed, the moduli of the double-covering tori are $\tau = (it/2 + 1/2)$ for the Möbius-strip amplitude, $\tau = 2it$ for the Klein-bottle amplitude and $\tau = it/2$ for the annulus amplitude. In Appendix B we give the definition of the characters used in Eqs. (3.3)–(3.5) in terms of $[SO(2)]^4$ characters.

It is worth to analyze the effects of the freely-acting operation on the geometry of the models. In general, $\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifolds contain O9-planes and three sets of O5$_i$-planes defined as the fixed tori of the operations $\Omega_P \circ g$, $\Omega_P \circ f$, $\Omega_P \circ h$, each wrapping one of the three internal tori T^i. In our freely-acting orbifold case, the overall O5$_i$-plane charges are zero and the O5$_i$-planes couple only to massive (odd-windings) states. A geometric picture of this fact can be obtained T-dualizing the two directions the O5$_i$ planes wrap, so that they become O3$_i$-planes. In this way, the freely acting operation replaces the O3$_{i,-}$ planes by (O3$_{i,+}$–O3$_{i,-}$) pairs, separated by half the lattice spacing in the coordinate affected by the free action. Since there are no global background charges from O5$_i$-planes, the model contains only background D9 branes. Finally, the Chan–Paton D9 charges are defined as,

$$\begin{aligned} I_N &= n_o + n_g + n_h + n_f, & g_N &= n_o + n_g - n_h - n_f, \\ h_N &= n_o - n_g + n_h - n_f, & f_N &= n_o - n_g - n_h + n_f, \end{aligned} \tag{3.8}$$

with $I_N = 32$ fixed by the tadpole cancellation condition. The massless spectrum has $\mathcal{N} = 1$ supersymmetry. The gauge group is $SO(n_o)\otimes SO(n_g)\otimes SO(n_h)\otimes SO(n_f)$, with chiral multiplets in the bifundamental representations

$$\begin{aligned} &(\mathbf{n_o}, \mathbf{n_g}, \mathbf{1}, \mathbf{1}) + (\mathbf{n_o}, \mathbf{1}, \mathbf{n_f}, \mathbf{1}) + (\mathbf{n_o}, \mathbf{1}, \mathbf{1}, \mathbf{n_h}) + (\mathbf{1}, \mathbf{n_g}, \mathbf{n_f}, \mathbf{1}) \\ &\quad + (\mathbf{1}, \mathbf{n_g}, \mathbf{1}, \mathbf{n_h}) + (\mathbf{1}, \mathbf{1}, \mathbf{n_f}, \mathbf{n_h}). \end{aligned} \tag{3.9}$$

The existence of four different Chan–Paton charges can be traced to the various consistent actions of the orbifold group on the Chan–Paton space or, alternatively, to the number of independent sectors of the chiral conformal field theory. It can be useful for the reader to make a connection with the alternative notation of [23]. The original Chan–Paton charges can be grouped into a 32×32 matrix λ. In this Chan–Paton matrix space, the three orbifold operations g, f and h act via matrices $\gamma_g, \gamma_f, \gamma_h$ which, correspondingly to (3.8), are given by

$$\gamma_g = (I_{n_o}, I_{n_g}, -I_{n_f}, -I_{n_h}),$$

[6] In what follows we set the string tension $\alpha' = 1/2$.

$$\gamma_f = (I_{n_o}, -I_{n_g}, I_{n_f}, -I_{n_h}),$$
$$\gamma_h = (I_{n_o}, -I_{n_g}, -I_{n_f}, I_{n_h}), \tag{3.10}$$

where I_{n_o} denote the identity matrix in the $n_o \times n_o$ block diagonal Chan–Paton matrix, and the same for the other multiplicities n_i. For $n_g = n_h = n_f = 0$ one recovers a pure $SO(32)$ SYM with no extra multiplets, a theory where gaugino condensation is expected to arise. Finally, let us notice that even if perturbatively n_o, n_g, n_f, n_h can be arbitrary positive integers subject only to the tadpole condition $n_o + n_g + n_f + n_h = 32$, non-perturbative consistency asks all of them to be even integers.

3.2. Type I racetrack model

In a variation of the previous $SO(32)$ model, we may add a discrete deformation along one of the unshifted directions, similar to a Wilson line $A_2 = (e^{2\pi i \mathbf{a}})$ along y^2, with $\mathbf{a} = (\mathbf{0}_p, \mathbf{1/2}_{32-p})$ and breaking $SO(32) \to SO(p) \otimes SO(32-p)$. The annulus and Möbius amplitudes, (3.4) and (3.5), get correspondingly modified to the following expressions:

$$\begin{aligned}
A = \int_0^\infty \frac{dt}{t^3\eta^4}\frac{1}{8}\bigg\{ & \big[\big(p^2+q^2\big)P_{m_1'} + 2pq\,P_{m_1'+\frac{1}{2}}\big]P_{m_1}P_2P_3(\tau_{oo}+\tau_{og}+\tau_{oh}+\tau_{of}) \\
& + \big(p^2+q^2\big)\big[(-1)^{m_2}P_2(\tau_{oo}-\tau_{og}-\tau_{oh}+\tau_{of}) \\
& + (-1)^{m_3}P_3(\tau_{oo}-\tau_{og}+\tau_{oh}-\tau_{of})\big]\frac{4\eta^2}{\vartheta_2^2} \\
& + (-1)^{m_1}\big[\big(p^2+q^2\big)P_{m_1'} + 2pq\,P_{m_1'+\frac{1}{2}}\big]P_{m_1}(\tau_{oo}+\tau_{og}-\tau_{oh}-\tau_{of})\frac{4\eta^2}{\vartheta_2^2}\bigg\},
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
M = -\frac{p+q}{8}\bigg\{ & P_1P_2P_3(\hat\tau_{oo}+\hat\tau_{og}+\hat\tau_{oh}+\tau_{of}) \\
& + (-1)^{m_1}P_1(\hat\tau_{oo}+\hat\tau_{og}-\hat\tau_{oh}-\hat\tau_{of})\frac{4\eta^2}{\vartheta_2^2} \\
& + (-1)^{m_2}P_2(\hat\tau_{oo}-\hat\tau_{og}-\hat\tau_{oh}+\hat\tau_{of})\frac{4\eta^2}{\vartheta_2^2} \\
& + (-1)^{m_3}P_3(\hat\tau_{oo}-\hat\tau_{og}+\hat\tau_{oh}-\hat\tau_{of})\frac{4\eta^2}{\vartheta_2^2}\bigg\}.
\end{aligned} \tag{3.12}$$

As mentioned, $I_N = p + q = 32$,

$$P_{m_1'}P_{m_1} \equiv P_1, \quad \text{and} \tag{3.13}$$

$$P_{m_1'+\frac{1}{2}}P_{m_1} \equiv \sum_{m,m'} \exp\bigg[-\frac{\pi t}{(\operatorname{Re}T_1)(\operatorname{Re}U_1)}\big|m' - iU_1 m + 1/2\big|^2\bigg]. \tag{3.14}$$

Hence, the resulting $SO(p) \otimes SO(32-p)$ gauge group is accompanied by a pure $\mathcal{N}=1$ SYM theory on both factors, leading to a racetrack scenario with two gaugino condensates. Indeed, in the four-dimensional effective supergravity Lagrangian, the tree-level gauge kinetic functions on the two stacks of D9 branes are equal,

$$f_{SO(p)} = f_{SO(q)} = S, \tag{3.15}$$

where S is the universal dilaton–axion chiral multiplet. Gaugino condensation on both stacks then generates the non-perturbative superpotential

$$W_{np} = A_p^{(k)} e^{-a_p S} + A_q^{(l)} e^{-a_q S}, \tag{3.16}$$

where $A_p^{(k)} = (p-2)\exp(2\pi i k/(p-2))$ and $A_q^{(l)} = (q-2)\exp(2\pi i l/(q-2))$, with $k = 1, \ldots, p-2$ and $l = 1, \ldots, q-2$, provide the requested different phases of the SYM vacua [24]. Moreover, $a_p = 2/(p-2)$ ($a_q = 2/(q-2)$) is related to the one-loop beta function of the $SO(p)$ ($SO(q)$) SYM gauge factor. In addition to the massless states, the model contains massive states, in particular a massive vector multiplet in the $(\mathbf{p}, \mathbf{q})$ bifundamental representation, with a lowest mass of the order of the compactification scale $M_c \sim 1/R$. Since the four-dimensional effective theory is valid anyway below M_c, these states are heavy and their effects on the low-energy physics can be encoded in threshold effects which we shall compute later on.

An interesting question is the geometrical interpretation of the present model.[7] The natural interpretation is in terms of a Wilson line breaking of the $SO(p) \otimes SO(32-p)$ model. The absence of scalars describing positions of the branes corresponding to each SO factor indicates that the corresponding branes are fractional and, as such, cannot move outside the fixed points. However, by giving vev's to the scalars in bifundamentals, one converts fractional branes into regular branes. The resulting gauge group is $SO(2P) \otimes SO(16-P)$, where the first factor comes from the branes sitting at the fixed point, while the second factor describe brane pairs in the bulk having scalars in the symmetric representation corresponding to their positions. Moving the bulk branes to another fixed-point, one gets, as usual, an enhancement of the gauge group to $SO(2P) \otimes SO(32-2P)$.

3.3. Type I with unitary groups

It is interesting to analyze the non-perturbative dynamics of the gauge theory on the D9 branes in the case of an orbifold action on the Chan–Paton space that produces unitary gauge groups. This can be done in a very simple way by choosing a different Chan–Paton assignment compared to (3.8). Consider the same cylinder amplitude (3.4) equipped with the following parametrization of the Chan–Paton charges:

$$\begin{aligned} I_N &= n + \bar{n} + m + \bar{m}, \qquad g_N = n + \bar{n} - m - \bar{m}, \\ f_N &= i(n - \bar{n} + m - \bar{m}), \qquad h_N = i(n - \bar{n} - m + \bar{m}). \end{aligned} \tag{3.17}$$

The Möbius amplitude has to be changed for consistency into

$$M = -\int_0^\infty \frac{dt}{t^3\eta^4} \frac{I_N}{8} \bigg\{ (\hat{\tau}_{oo} + \hat{\tau}_{og} + \hat{\tau}_{oh} + \hat{\tau}_{of}) P_1 P_2 P_3 \\ + (\hat{\tau}_{oo} + \hat{\tau}_{og} - \hat{\tau}_{oh} - \hat{\tau}_{of}) (-1)^{m_1} P_1 \frac{4\hat{\eta}^2}{\hat{\vartheta}_2^2}$$

[7] E.D. is grateful to C. Angelantonj and M. Bianchi for illuminating discussions on this and the other string models presented in the present paper.

$$- (\hat{\tau}_{oo} - \hat{\tau}_{og} + \hat{\tau}_{oh} - \hat{\tau}_{of})(-1)^{m_3} P_3 \frac{4\hat{\eta}^2}{\hat{\vartheta}_2^2}$$
$$- (\hat{\tau}_{oo} - \hat{\tau}_{og} - \hat{\tau}_{oh} + \hat{\tau}_{of})(-1)^{m_2} P_2 \frac{4\hat{\eta}^2}{\hat{\vartheta}_2^2} \Bigg\}, \tag{3.18}$$

where the changes of sign in the D9–O5$_2$ and D9–O5$_3$ propagation, needed to enforce the unitary projection, are interpreted as discrete Wilson lines on the D9 branes in the last two torii [20]. The massless open string amplitudes,

$$\begin{aligned} A_0 + M_0 = {} & (n\bar{n} + m\bar{m})\tau_{oo} + \left[\frac{n(n-1)}{2} + \frac{\bar{n}(\bar{n}-1)}{2} + \frac{m(m-1)}{2} + \frac{\bar{m}(\bar{m}-1)}{2}\right]\tau_{og} \\ & + (n\bar{m} + \bar{n}m)\tau_{of} + (nm + \bar{n}\bar{m})\tau_{oh}, \end{aligned} \tag{3.19}$$

exhibit the spectrum of an $\mathcal{N} = 1$ supersymmetric $U(n) \otimes U(m)$ theory, with $n + m = 16$ due to the (D9/O9) RR tadpole cancellation condition. Matter fields fall into massless chiral multiplets in the representations

$$\begin{aligned} & \left(\frac{\mathbf{n(n-1)}}{\mathbf{2}} + \frac{\mathbf{\bar{n}(\bar{n}-1)}}{\mathbf{2}}, \mathbf{1}\right) + \left(\mathbf{1}, \frac{\mathbf{m(m-1)}}{\mathbf{2}} + \frac{\mathbf{\bar{m}(\bar{m}-1)}}{\mathbf{2}}\right) \\ & + (\mathbf{n}, \mathbf{\bar{m}}) + (\mathbf{\bar{n}}, \mathbf{m}) + (\mathbf{n}, \mathbf{m}) + (\mathbf{\bar{n}}, \mathbf{\bar{m}}). \end{aligned} \tag{3.20}$$

Notice that the choice $m = 0$ with a gauge group $U(16)$, in contrast to the $SO(32)$ case, is not pure SYM, since it contains massless chiral multiplets in the $(\mathbf{120} + \overline{\mathbf{120}})$ representation.

The gauge theory on D9 branes is not really supersymmetric QCD with flavors in the fundamental and antifundamental representation, whose non-perturbative dynamics is known with great accuracy [9]. One way to get a more interesting example is the following. Moving p D9 branes out of the total 16 to a different orientifold fixed point not affected by the shift, one gets a gauge group $U(n) \otimes U(m) \otimes U(p)$, with $n + m + p = 16$. Strings stretched between the p D9 branes and the remaining $n + m$ are massive, and therefore they disappear from the effective low-energy gauge theory, whereas the $U(n) \otimes U(m)$ gauge sector has the massless spectrum displayed in (3.20). Choosing $n = 3$ and $m = 1$, a gauge group $SU(3) \otimes U(1)^2$ results, together with a factor $U(12)$ decoupled from it. Using the fact that the antisymmetric representation of $SU(3)$ coincides with the antifundamental $\bar{3}$, one ends up with an SQCD theory with gauge group $SU(3)$ and $N_f = 3$ flavors of quarks–antiquarks. This is the regime $N_c = N_f = N$ described in [25], where the composite mesons $M = Q\bar{Q}$ and baryons (antibaryons) $B = Q_1 \cdots Q_n$ ($\tilde{B} = \tilde{Q}_1 \cdots \tilde{Q}_n$) have a quantum-deformed moduli space such that

$$\det M - B\tilde{B} = \Lambda^{2N}, \tag{3.21}$$

where $\Lambda^{2N} = \exp(-8\pi^2/g^2)$ is the dynamical scale of the $SU(3)$ gauge theory. As a consequence, the deformation in (3.21) originates only from the one-instanton contribution.

4. Heterotic dual models

4.1. Heterotic $SO(32)$ model

Due to the freely-acting nature of the type I orbifold, according to the adiabatic argument [12] the S-duality between the type I and the $SO(32)$ heterotic string is expected to be preserved.

In this section we explicitly construct the heterotic S-dual of the $SO(32)$ type I model.[8] The natural guess is to use the same freely-acting orbifold generators with a trivial action on the internal gauge degrees of freedom, consistently with the fact that in its type I dual the action on the Chan–Paton factors is trivial as well. There is however one subtlety, already encountered in similar situations and explained in other examples in [12]. Modular invariance forces us to change the geometric freely-orbifold actions (2.8)–(2.10) into a non-geometric one. Let us consider for simplicity one circle of radius R and one of the geometric shift in (2.8)–(2.10)

$$X \to X + \pi R. \tag{4.1}$$

Our claim is that its S-dual on the heterotic side is the non-geometric action[9]

$$X_L \to X_L + \frac{\pi R}{2} + \frac{\pi\alpha'}{2R}, \qquad X_R \to X_R + \frac{\pi R}{2} - \frac{\pi\alpha'}{2R}. \tag{4.2}$$

In order to prove this claim, we use the fermionic formulation of the sixteen-dimensional heterotic gauge lattice, with 16 complex fermions. Guided by the type I dual model, we take a trivial orbifold action on the 16 gauge fermions. The adiabatic argument of [12] allows identification of the orbifold action only in the large radius limit, where the shift (4.2) is indistinguishable from (4.1). In the twisted sector of the theory, the masses of the lattice states (m,n) are shifted according to

$$(m,n) \to (m+s_1, n+s_1'), \tag{4.3}$$

where $(s_1,s_1') = (1/2,0)$ for (4.1) and $(s_1,s_1') = (1/2,1/2)$ for (4.2). The Virasoro generators of the left and right CFT's are

$$\begin{aligned} L_0 &= N + 2\times\left(-\frac{1}{12} - \frac{1}{24}\right) + 2\times\left(\frac{1}{24} + \frac{1}{12}\right), \\ \bar{L}_0 &= \tilde{N} + 10\times\left(-\frac{1}{12}\right) + 2\times\frac{1}{24}, \end{aligned} \tag{4.4}$$

where N ($\tilde{N}$) contains the oscillator contributions whereas the other terms are the zero-point energy in the NS sector from the spacetime and the gauge coordinates. Level-matching in the twisted sector is then

$$L_0 - \bar{L}_0 = N - \tilde{N} + \frac{3}{4} = -(m+s_1)(n+s_1') \ (\mathrm{mod}\ 1). \tag{4.5}$$

This is possible only for $(s_1,s_1') = (1/2,1/2)$ which therefore fixes (4.2) to be the correct choice. The S-dual of the type I freely-acting $SO(32)$ is then defined by the modular invariant torus amplitude

$$\begin{aligned} T = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^3\eta^2\bar{\eta}^2}\frac{1}{4}\bigg[&(\tau_{oo}+\tau_{og}+\tau_{oh}+\tau_{of})\Lambda_1\Lambda_2\Lambda_3 \\ &+ (\tau_{oo}+\tau_{og}-\tau_{oh}-\tau_{of})(-1)^{m_1+n_1}\Lambda_1\left|\frac{4\eta^2}{\vartheta_2^2}\right|^2 \end{aligned}$$

[8] We are grateful to M. Bianchi and E. Kiritsis for helpful discussions and comments on this point.

[9] As shown recently [26], such asymmetric shifts in type I models are consistent only if they act in an even number of coordinates.

$$
\begin{aligned}
&+ (\tau_{oo} - \tau_{og} + \tau_{oh} - \tau_{of})(-1)^{m_3+n_3} \Lambda_3 \left|\frac{4\eta^2}{\vartheta_2^2}\right|^2 \\
&+ (\tau_{oo} - \tau_{og} - \tau_{oh} + \tau_{of})(-1)^{m_2+n_2} \Lambda_2 \left|\frac{4\eta^2}{\vartheta_2^2}\right|^2 \\
&+ (\tau_{go} + \tau_{gg} + \tau_{gh} + \tau_{gf}) \Lambda_1^{m_1+\frac{1}{2},n_1+\frac{1}{2}} \left|\frac{4\eta^2}{\vartheta_4^2}\right|^2 \\
&+ (\tau_{ho} + \tau_{hg} + \tau_{hh} + \tau_{hf}) \Lambda_3^{m_3+\frac{1}{2},n_3+\frac{1}{2}} \left|\frac{4\eta^2}{\vartheta_4^2}\right|^2 \\
&+ (\tau_{fo} + \tau_{fg} + \tau_{fh} + \tau_{ff}) \Lambda_2^{m_2+\frac{1}{2},n_2+\frac{1}{2}} \left|\frac{4\eta^2}{\vartheta_4^2}\right|^2 \\
&- (\tau_{go} + \tau_{gg} - \tau_{gh} - \tau_{gf})(-1)^{m_1+n_1} \Lambda_1^{m_1+\frac{1}{2},n_1+\frac{1}{2}} \left|\frac{4\eta^2}{\vartheta_3^2}\right|^2 \\
&- (\tau_{ho} - \tau_{hg} + \tau_{hh} - \tau_{hf})(-1)^{m_3+n_3} \Lambda_3^{m_3+\frac{1}{2},n_3+\frac{1}{2}} \left|\frac{4\eta^2}{\vartheta_3^2}\right|^2 \\
&- (\tau_{fo} - \tau_{fg} - \tau_{fh} + \tau_{ff})(-1)^{m_2+n_2} \Lambda_2^{m_2+\frac{1}{2},n_2+\frac{1}{2}} \left|\frac{4\eta^2}{\vartheta_3^2}\right|^2 \Bigg] \times (\bar{O}_{32} + \bar{S}_{32}). \qquad (4.6)
\end{aligned}
$$

Indeed, the massless spectrum matches perfectly with its type I counterpart. Compared to its type I S-dual cousin, the heterotic model has the same spectrum for the Kaluza–Klein modes, whereas it has a different spectrum for the winding modes. This is precisely what is expected from S-duality [11], which maps KK states into KK states, whereas it maps perturbative winding states into non-perturbative states in the S-dual theory.

4.2. Dual heterotic models with orthogonal gauge groups

In the fermionic formulation, the dual of the type I $SO(n_o) \otimes SO(n_g) \otimes SO(n_h) \otimes SO(n_f)$, $n_0 + n_g + n_f + n_h = 32$ can be constructed by splitting the 16 complex fermions of the gauge lattice into $n_0/2 + n_g/2 + n_f/2 + n_h/2$ groups. We then embed the orbifold action into the gauge lattice as shown in Table 1.

Level matching in this case can be readily worked out with the result, in the g, f and h twisted sectors respectively

$$
\begin{aligned}
L_0 - \bar{L}_0 &= N - \tilde{N} - \frac{5}{4} + \frac{n_o + n_g}{16} - (m_1 + s_1)(n_1 + s_1') \ (\mathrm{mod}\ 1), \\
L_0 - \bar{L}_0 &= N - \tilde{N} - \frac{5}{4} + \frac{n_o + n_f}{16} - (m_2 + s_2)(n_2 + s_2') \ (\mathrm{mod}\ 1), \\
L_0 - \bar{L}_0 &= N - \tilde{N} - \frac{5}{4} + \frac{n_o + n_h}{16} - (m_3 + s_3)(n_3 + s_3') \ (\mathrm{mod}\ 1). \qquad (4.7)
\end{aligned}
$$

The various possibilities are then as follows

- $n_o + n_g = 8 \ (\mathrm{mod}\ 8) \to s_1 = s_1' = 1/2$,
- $n_o + n_g = 4 \ (\mathrm{mod}\ 8) \to s_1 = 1/2,\ s_1' = 0$,

Table 1
Orbifold actions in the gauge degrees of freedom in the fermionic formulation

Orb. actions	$SO(n_o)$	$SO(n_g)$	$SO(n_f)$	$SO(n_h)$
g	$+$	$+$	$-$	$-$
f	$+$	$-$	$+$	$-$
h	$+$	$-$	$-$	$+$

and similarly for the other pairs $n_o + n_f$, $n_o + n_h$. It is interesting to notice the restrictions on the rank of the gauge group. While the restriction on the even $SO(2n)$ gauge factors was expected from the beginning, the above conditions are actually stronger.

Let us take a closer look to the particular case of the gauge group $SO(p) \otimes SO(q)$ with $p + q = 32$, in order to better understand this point. The corresponding setting is $n_o = p$, $n_g = q$ and $n_f = n_h = 0$. Level matching in the f and h twisted sectors reads

$$L_0 - \bar{L}_0 = N - \tilde{N} - \frac{5}{4} + \frac{p}{16} = -(m + s_1)(n + s_2) \ (\mathrm{mod}\ 1), \tag{4.8}$$

which leads to the following options:

- $p = 8 \ (\mathrm{mod}\ 8) \to s_1 = s_2 = 1/2$,
- $p = 4 \ (\mathrm{mod}\ 8) \to s_1 = 1/2$, $s_2 = 0$.

Surprisingly, we do not find solutions for $p = 2 \ (\mathrm{mod}\ 2)$. We can only speculate that, perhaps, a more subtle orbifold actions on the gauge lattice and/or the introduction of discrete Wilson lines could help in finding the $p = 2$ models, which the dual type I models suggest that have to exist.

For the first case, $p = 8, 16, 24$, it is convenient, in the fermionic formulation of the gauge degrees of freedom, to define the following characters

$$\begin{aligned} \chi_o &= O_p O_q + C_p C_q, & \chi_v &= V_p V_q + S_p S_q, \\ \chi_s &= O_p C_q + C_p O_q, & \chi_c &= V_p S_q + S_p V_q. \end{aligned} \tag{4.9}$$

The complete partition function of the heterotic model is then

$$\begin{aligned} T = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^3 \eta^2 \bar{\eta}^2} \frac{1}{4} \Bigg\{ \Bigg[& (\tau_{oo} + \tau_{og} + \tau_{oh} + \tau_{of}) \Lambda_1 \Lambda_2 \Lambda_3 \\ & + (\tau_{oo} + \tau_{og} - \tau_{oh} - \tau_{of}) (-1)^{m_1+n_1} \Lambda_1 \left| \frac{4\eta^2}{\vartheta_2^2} \right|^2 \\ & + (\tau_{go} + \tau_{gg} + \tau_{gh} + \tau_{gf}) \Lambda_1^{m_1+\frac{1}{2}, n_1+\frac{1}{2}} \left| \frac{4\eta^2}{\vartheta_4^2} \right|^2 \\ & + (\tau_{go} + \tau_{gg} - \tau_{gh} - \tau_{gf}) (-1)^{m_1+n_1} \Lambda_1^{m_1+\frac{1}{2}, n_1+\frac{1}{2}} \left| \frac{4\eta^2}{\vartheta_3^2} \right|^2 \Bigg] (\overline{\chi_o + \chi_v}) \\ & + \Big[(\tau_{oo} - \tau_{og} + \tau_{oh} - \tau_{of}) (-1)^{m_3+n_3} \Lambda_3 \\ & + (\tau_{oo} - \tau_{og} - \tau_{oh} + \tau_{of}) (-1)^{m_2+n_2} \Lambda_2 \Big] \left| \frac{4\eta^2}{\vartheta_2^2} \right|^2 (\overline{\chi_o - \chi_v}) \end{aligned}$$

$$+\left[(\tau_{ho}+\tau_{hg}+\tau_{hh}+\tau_{hf})\Lambda_3^{m_3+\frac{1}{2},n_3+\frac{1}{2}}\right.$$
$$\left.+(\tau_{fo}+\tau_{fg}+\tau_{fh}+\tau_{ff})\Lambda_2^{m_2+\frac{1}{2},n_2+\frac{1}{2}}\right]\left|\frac{4\eta^2}{\vartheta_4^2}\right|^2(\overline{\chi_s+\chi_c})$$
$$-(-1)^{q/8}\left[(\tau_{ho}-\tau_{hg}+\tau_{hh}-\tau_{hf})(-1)^{m_3+n_3}\Lambda_3^{m_3+\frac{1}{2},n_3+\frac{1}{2}}\right.$$
$$\left.\left.+(\tau_{fo}-\tau_{fg}-\tau_{fh}+\tau_{ff})(-1)^{m_2+n_2}\Lambda_2^{m_2+\frac{1}{2},n_2+\frac{1}{2}}\right]\left|\frac{4\eta^2}{\vartheta_3^2}\right|^2(\overline{\chi_s-\chi_c})\right\}. \tag{4.10}$$

As for the $SO(32)$ model, the whole KK spectrum precisely match the corresponding one on the type I S-dual side, whereas the massive winding states and the massive twisted spectra are, as expected, quite different. On the other hand, for the second case $p = 4, 12, 20$, the correct characters are

$$\chi_o = O_pO_q + C_pC_q, \qquad \chi_v = V_pV_q + S_pS_q,$$
$$\chi_s = V_pC_q + S_pO_q, \qquad \chi_c = O_pS_q + C_pV_q. \tag{4.11}$$

The complete partition function is now

$$T=\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2^3\eta^2\bar{\eta}^2}\frac{1}{4}\left\{\left[(\tau_{oo}+\tau_{og}+\tau_{oh}+\tau_{of})\Lambda_1\Lambda_2\Lambda_3\right.\right.$$
$$+(\tau_{oo}+\tau_{og}-\tau_{oh}-\tau_{of})(-1)^{m_1+n_1}\Lambda_1\left|\frac{4\eta^2}{\vartheta_2^2}\right|^2$$
$$+(\tau_{go}+\tau_{gg}+\tau_{gh}+\tau_{gf})\Lambda_1^{m_1+\frac{1}{2},n_1+\frac{1}{2}}\left|\frac{4\eta^2}{\vartheta_4^2}\right|^2$$
$$\left.+(\tau_{go}+\tau_{gg}-\tau_{gh}-\tau_{gf})(-1)^{m_1+n_1}\Lambda_1^{m_1+\frac{1}{2},n_1+\frac{1}{2}}\left|\frac{4\eta^2}{\vartheta_3^2}\right|^2\right](\overline{\chi_o+\chi_v})$$
$$+\left[(\tau_{oo}-\tau_{og}+\tau_{oh}-\tau_{of})(-1)^{m_3}\Lambda_3\right.$$
$$\left.+(\tau_{oo}-\tau_{og}-\tau_{oh}+\tau_{of})(-1)^{m_2}\Lambda_2\right]\left|\frac{4\eta^2}{\vartheta_2^2}\right|^2(\overline{\chi_o-\chi_v})$$
$$+\left[(\tau_{ho}+\tau_{hg}+\tau_{hh}+\tau_{hf})\Lambda_3^{m_3,n_3+\frac{1}{2}}\right.$$
$$\left.+(\tau_{fo}+\tau_{fg}+\tau_{fh}+\tau_{ff})\Lambda_2^{m_2,n_2+\frac{1}{2}}\right]\left|\frac{4\eta^2}{\vartheta_4^2}\right|^2(\overline{\chi_s+\chi_c})$$
$$-(-1)^{(p+4)/8}\left[(\tau_{ho}-\tau_{hg}+\tau_{hh}-\tau_{hf})(-1)^{m_3}\Lambda_3^{m_3,n_3+\frac{1}{2}}\right.$$
$$\left.\left.+(\tau_{fo}-\tau_{fg}-\tau_{fh}+\tau_{ff})(-1)^{m_2}\Lambda_2^{m_2,n_2+\frac{1}{2}}\right]\left|\frac{4\eta^2}{\vartheta_3^2}\right|^2(\overline{\chi_s-\chi_c})\right\}. \tag{4.12}$$

It should be noticed that while the KK spectra are actually the same for the two cases $p = 4$ and $p = 8 \pmod 8$, they are very different in the massive winding sector, in perfect agreement with the modular invariance constraints (4.7).

We expect that appropriate orbifold action in the sixteen-dimensional gauge lattice will also produce the S-dual of the type I racetrack and of the unitary gauge group cases, discussed in the previous sections. The required action, however, cannot correspond to a standard Wilson line in the adjoint of the gauge group, but rather to a non-diagonal action in the Cartan basis, like the ones considered in [27].

5. Threshold corrections to the gauge couplings

In this section we perform the one-loop calculation of the threshold corrections to the gauge couplings of some of the models described in the previous sections. The effective field theory quantities can be then easily extracted from the one-loop computation.

The threshold correction Λ_2 is generically written as

$$\left.\frac{4\pi^2}{g_a^2}\right|_{\text{1-loop}} = \left.\frac{4\pi^2}{g_a^2}\right|_{\text{tree}} + \Lambda_{2,a}, \tag{5.1}$$

with

$$\Lambda_{2,a} = \int_{\mathcal{F}} \frac{d^2\tau}{4\tau_2}\mathcal{B}_a(\tau) \tag{5.2}$$

for the heterotic string, and

$$\Lambda_{2,a} = \int_0^\infty \frac{dt}{4t}\mathcal{B}_a(t) \tag{5.3}$$

for the type I string. In these expressions, $\mathcal{B}_a$ flows in the infrared to

$$b_a = -3T_a(G) + \sum_r T_a(r), \tag{5.4}$$

the one-loop beta function for the gauge group factor G_a, with r running over the gauge group representations with Dynkin index $T_a(r)$. From the one-loop expression of the gauge coupling it is possible to extract [30] the holomorphic gauge couplings $f_a(M_i)$, where M_i denote here collectively the moduli chiral (super)fields, using the relation [31]

$$\begin{aligned}\frac{4\pi^2}{g_a^2(\mu^2)} = {} & \operatorname{Re} f_a + \frac{b_a}{4}\log\frac{M_P^2}{\mu^2} + \frac{c_a}{4}K + \frac{T_a(G)}{2}\ln g_a^{-2}\left(\mu^2\right) \\ & - \sum_r \frac{T_a(r)}{2}\ln\det Z_r(\mu)^2,\end{aligned} \tag{5.5}$$

where K is the Kähler potential, Z_r is the wave-function normalization matrix for the matter fields and $c_a = \sum_r T_a(r) - T_a(G)$. With this definition, the holomorphic non-perturbative scale Λ_a of an asymptotically-free gauge theory ($b_a < 0$) is given by

$$\Lambda_a = M_P e^{-\frac{2f_a}{|b_a|}}. \tag{5.6}$$

5.1. Type I $SO(n_o) \otimes SO(n_g) \otimes SO(n_f) \otimes SO(n_h)$ model

For the computation of threshold corrections to the gauge couplings in the freely-acting type I model with orthogonal gauge groups, we make use of the background field method [28–30].

Therefore, we introduce a magnetic field along two of the spatial non-compact directions, say $F_{23} = BQ$. In the weak field limit, the one-loop vacuum energy can be expanded in powers of B, providing

$$\Lambda(B) = \Lambda_0 + \frac{1}{2}\left(\frac{B}{2\pi}\right)^2 \Lambda_2 + \cdots. \tag{5.7}$$

For supersymmetric vacua $\Lambda_0 = 0$, and the quadratic term accounts exactly for the threshold corrections in Eq. (5.1).

In the presence of F_{23}, the oscillator modes along the non-compact complex plane $x^2 + ix^3$ get shifted by an amount ϵ such that

$$\pi\epsilon = \arctan(\pi q_L B) + \arctan(\pi q_R B) \simeq \pi(q_L + q_R)B + O\big(B^3\big), \tag{5.8}$$

where q_L and q_R are the eigenvalues of the gauge group generator Q, acting on the Chan–Paton states localized at the two endpoints of the open strings. In the vacuum energy, the contribution of the non-compact bosons and fermions gets replaced by

$$\frac{\vartheta_\alpha(0|\tau)}{\eta^3(\tau)} \to 2\pi\epsilon\tau \frac{\vartheta_\alpha(\tau\epsilon|\tau)}{\vartheta_1(\tau\epsilon|\tau)} \quad \text{for } \alpha = 2, 3, 4 \tag{5.9}$$

in the annulus and Möbius amplitudes. In addition, the momentum operator along the non-compact dimensions becomes,

$$p^\mu p_\mu \to -(p_0)^2 + (p_1)^2 + (2n+1)\epsilon + 2\epsilon\Sigma_{23}, \tag{5.10}$$

where Σ_{23} is the spin operator in the (23) direction, while n is an integer that labels the Landau levels. The supertrace operator becomes now

$$\mathrm{STr} \to \left(\sum_{\text{bos}} - \sum_{\text{ferm}}\right) \frac{(q_L + q_R)B}{2\pi} \int \frac{d^2p}{(2\pi)^2}, \tag{5.11}$$

where $(q_L + q_R)B/2\pi$ is the density of the Landau levels and the integral is performed only over the momenta in the non-compact directions x^0 and x^1.

The details of the computation can be found in Appendix C.1. Collecting the results obtained there, and assuming Q to be in a $U(1)$ inside $SO(n_o)$, $SO(n_g)$, $SO(n_f)$ or $SO(n_h)$, the moduli dependent threshold corrections for the respective gauge couplings can be written as follows,

$$\begin{aligned}
\Lambda_{2,o} = -\frac{1}{4}\mathrm{Tr}\big(Q^2\big)\bigg[&(2-g_N)\bigg(\pi \operatorname{Re} U_1 + \log\bigg[(\operatorname{Re} U_1)(\operatorname{Re} T_1)\mu^2\bigg|\frac{\vartheta_4}{\eta^3}(2iU_1)\bigg|^{-2}\bigg]\bigg) \\
&+ (2-f_N)\bigg(\pi \operatorname{Re} U_2 + \log\bigg[(\operatorname{Re} U_2)(\operatorname{Re} T_2)\mu^2\bigg|\frac{\vartheta_4}{\eta^3}(2iU_2)\bigg|^{-2}\bigg]\bigg) \\
&+ (2-h_N)\bigg(\pi \operatorname{Re} U_3 + \log\bigg[(\operatorname{Re} U_3)(\operatorname{Re} T_3)\mu^2\bigg|\frac{\vartheta_4}{\eta^3}(2iU_3)\bigg|^{-2}\bigg]\bigg)\bigg],
\end{aligned} \tag{5.12}$$

$$\begin{aligned}
\Lambda_{2,g} = -\frac{1}{4}\mathrm{Tr}\big(Q^2\big)\bigg[&(2-g_N)\bigg(\pi \operatorname{Re} U_1 + \log\bigg[(\operatorname{Re} U_1)(\operatorname{Re} T_1)\mu^2\bigg|\frac{\vartheta_4}{\eta^3}(2iU_1)\bigg|^{-2}\bigg]\bigg) \\
&+ (2+f_N)\bigg(\pi \operatorname{Re} U_2 + \log\bigg[(\operatorname{Re} U_2)(\operatorname{Re} T_2)\mu^2\bigg|\frac{\vartheta_4}{\eta^3}(2iU_2)\bigg|^{-2}\bigg]\bigg)
\end{aligned}$$

$$+ (2+h_N)\left(\pi \operatorname{Re} U_3 + \log\left[(\operatorname{Re} U_3)(\operatorname{Re} T_3)\mu^2\left|\frac{\vartheta_4}{\eta^3}(2iU_3)\right|^{-2}\right]\right)\Bigg], \tag{5.13}$$

$$\begin{aligned}\Lambda_{2,f} = -\frac{1}{4}\operatorname{Tr}(Q^2)\Bigg[&(2+g_N)\left(\pi \operatorname{Re} U_1 + \log\left[(\operatorname{Re} U_1)(\operatorname{Re} T_1)\mu^2\left|\frac{\vartheta_4}{\eta^3}(2iU_1)\right|^{-2}\right]\right)\\ &+ (2-f_N)\left(\pi \operatorname{Re} U_2 + \log\left[(\operatorname{Re} U_2)(\operatorname{Re} T_2)\mu^2\left|\frac{\vartheta_4}{\eta^3}(2iU_2)\right|^{-2}\right]\right)\\ &+ (2+h_N)\left(\pi \operatorname{Re} U_3 + \log\left[(\operatorname{Re} U_3)(\operatorname{Re} T_3)\mu^2\left|\frac{\vartheta_4}{\eta^3}(2iU_3)\right|^{-2}\right]\right)\Bigg],\end{aligned} \tag{5.14}$$

$$\begin{aligned}\Lambda_{2,h} = -\frac{1}{4}\operatorname{Tr}(Q^2)\Bigg[&(2+g_N)\left(\pi \operatorname{Re} U_1 + \log\left[(\operatorname{Re} U_1)(\operatorname{Re} T_1)\mu^2\left|\frac{\vartheta_4}{\eta^3}(2iU_1)\right|^{-2}\right]\right)\\ &+ (2+f_N)\left(\pi \operatorname{Re} U_2 + \log\left[(\operatorname{Re} U_2)(\operatorname{Re} T_2)\mu^2\left|\frac{\vartheta_4}{\eta^3}(2iU_2)\right|^{-2}\right]\right)\\ &+ (2-h_N)\left(\pi \operatorname{Re} U_3 + \log\left[(\operatorname{Re} U_3)(\operatorname{Re} T_3)\mu^2\left|\frac{\vartheta_4}{\eta^3}(2iU_3)\right|^{-2}\right]\right)\Bigg].\end{aligned} \tag{5.15}$$

The β-function coefficients can also be extracted in the form

$$\begin{aligned}b_o &= -\big[3(n_o-2) - (n_f+n_g+n_h)\big],\\ b_g &= -\big[3(n_g-2) - (n_f+n_o+n_h)\big],\\ b_f &= -\big[3(n_f-2) - (n_o+n_g+n_h)\big],\\ b_h &= -\big[3(n_h-2) - (n_f+n_g+n_o)\big],\end{aligned} \tag{5.16}$$

and, using the definition (5.5), the holomorphic one-loop gauge kinetic functions are then

$$\begin{aligned}f_o = S + \frac{1}{2}\Bigg[&(2-g_N)\log\frac{\vartheta_4}{e^{\pi U_1/2}\eta^3}(2iU_1) + (2-f_N)\log\frac{\vartheta_4}{e^{\pi U_2/2}\eta^3}(2iU_2)\\ &+ (2-h_N)\log\frac{\vartheta_4}{e^{\pi U_3/2}\eta^3}(2iU_3)\Bigg],\\ f_g = S + \frac{1}{2}\Bigg[&(2-g_N)\log\frac{\vartheta_4}{e^{\pi U_1/2}\eta^3}(2iU_1) + (2+f_N)\log\frac{\vartheta_4}{e^{\pi U_2/2}\eta^3}(2iU_2)\\ &+ (2+h_N)\log\frac{\vartheta_4}{e^{\pi U_3/2}\eta^3}(2iU_3)\Bigg],\\ f_f = S + \frac{1}{2}\Bigg[&(2+g_N)\log\frac{\vartheta_4}{e^{\pi U_1/2}\eta^3}(2iU_1) + (2-f_N)\log\frac{\vartheta_4}{e^{\pi U_2/2}\eta^3}(2iU_2)\\ &+ (2+h_N)\log\frac{\vartheta_4}{e^{\pi U_3/2}\eta^3}(2iU_3)\Bigg],\\ f_h = S + \frac{1}{2}\Bigg[&(2+g_N)\log\frac{\vartheta_4}{e^{\pi U_1/2}\eta^3}(2iU_1) + (2+f_N)\log\frac{\vartheta_4}{e^{\pi U_2/2}\eta^3}(2iU_2)\\ &+ (2-h_N)\log\frac{\vartheta_4}{e^{\pi U_3/2}\eta^3}(2iU_3)\Bigg].\end{aligned} \tag{5.17}$$

It is very important to stress the linear dependence of the above threshold corrections on the $(\pi \operatorname{Re} U_i)$ factors. Indeed, the presence of such terms in a loop contribution may seem surprising.

However, expanding the factor $\vartheta_4\eta^{-3}$, it can be realized that this term exactly cancels the contributions coming from the factor $q^{1/24}$ contained in the η-function. Thus, the total dependence on the moduli of the threshold corrections turns out to be exclusively of logarithmic form. This phenomenon can be physically understood making the observation that, beyond the Kaluza–Klein scale, $\mathcal{N}=4$ supersymmetry is effectively recovered. Therefore, in the large volume limit only logarithmic corrections in the moduli should be present. The price one has to pay is that modular invariance in the target space is lost, as evident from the above expressions. The breaking of modular invariance in the target space by the shift $\mathbb{Z}_2\times\mathbb{Z}_2$ orbifold is very different from what happens in the ordinary $\mathbb{Z}_2\times\mathbb{Z}_2$ case where, beyond the Kaluza–Klein scale, the effective supersymmetry for each sector is still $\mathcal{N}=2$. The threshold corrections in that case turn out to be proportional to $(\operatorname{Re}U)\log|\eta(iU)|^4$. Therefore, they preserve modular invariance, but have a non-logarithmic dependence on the moduli, due to the term $q^{1/24}$ inside the η-function.

5.2. *Type I racetrack model*

The details of the calculation can be found again in Appendix C.2. Using the background field method, the moduli dependent part of the gauge coupling threshold corrections is given by

$$\Lambda_{2,p}=-\frac{1}{4}\operatorname{Tr}\big(Q^2\big)\Bigg[(2-p)\sum_{j=1}^{3}\Bigg(\pi\operatorname{Re}U_j+\log\Bigg[(\operatorname{Re}U_j)(\operatorname{Re}T_j)\mu^2\bigg|\frac{\vartheta_4}{\eta^3}(2iU_j)\bigg|^{-2}\Bigg]\Bigg)$$
$$+q\Bigg(\log\bigg|\frac{\vartheta_4}{\eta^3}(2iU_1)\bigg|^2-\log\bigg|\frac{\vartheta_4}{\eta^3}(4iU_1)\bigg|^2+\pi\operatorname{Re}U_1\Bigg)\Bigg], \tag{5.18}$$

together with a similar expression for the $SO(q)$ factor, with the obvious replacements. The corresponding β-function coefficients of the $SO(p)$ and $SO(q)$ gauge group factors are

$$b_p=-3(p-2),\qquad b_q=-3(q-2), \tag{5.19}$$

and the one-loop holomorphic gauge functions read

$$f_p=S+\frac{2-p}{2}\sum_{i=1}^{3}\log\frac{\vartheta_4}{e^{\pi U_i/2}\eta^3}(2iU_i)-\frac{q}{2}\bigg[\log\frac{\vartheta_4}{e^{\pi U_1/2}\eta^3}(2iU_1)-\log\frac{\vartheta_4}{e^{\pi U_1}\eta^3}(4iU_1)\bigg],$$
$$f_q=S+\frac{2-q}{2}\sum_{i=1}^{3}\log\frac{\vartheta_4}{e^{\pi U_i/2}\eta^3}(2iU_i)-\frac{p}{2}\bigg[\log\frac{\vartheta_4}{e^{\pi U_1/2}\eta^3}(2iU_1)-\log\frac{\vartheta_4}{e^{\pi U_1}\eta^3}(4iU_1)\bigg].$$

The non-perturbative superpotential can be written, in analogy with (3.16),

$$W_{np}=A_p(U_i)e^{-a_pS}+A_q(U_i)e^{-a_qS}, \tag{5.20}$$

where

$$a_p=\frac{2}{p-2},\qquad A_p=\Bigg[\prod_{i=1}^{3}e^{-\pi U_i/2}\frac{\vartheta_4}{\eta^3}(2iU_i)\Bigg]\Bigg[e^{\pi U_1/2}\frac{\vartheta_4}{\eta^3}(2iU_1)\frac{\eta^3}{\vartheta_4}(4iU_1)\Bigg]^{\frac{q}{p-2}},$$
$$a_q=\frac{2}{q-2},\qquad A_q=\Bigg[\prod_{i=1}^{3}e^{-\pi U_i/2}\frac{\vartheta_4}{\eta^3}(2iU_i)\Bigg]\Bigg[e^{\pi U_1/2}\frac{\vartheta_4}{\eta^3}(2iU_1)\frac{\eta^3}{\vartheta_4}(4iU_1)\Bigg]^{\frac{p}{q-2}}. \tag{5.21}$$

5.3. Heterotic SO(32) model

For the heterotic string, several procedures are available in literature to extract the threshold corrections [32–34]. The general expression for the threshold corrections to the gauge couplings, valid in the $\overline{\rm DR}$ renormalization scheme, is given by

$$\Lambda_{2,a} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \frac{i}{4\pi} \frac{1}{|\eta|^2} \sum_{\alpha,\beta=0,1/2} \partial_\tau \left(\frac{\vartheta\left[{\alpha \atop \beta}\right]}{\eta} \right) \left(Q_a^2 - \frac{1}{4\pi\tau_2} \right) C\left[{\alpha \atop \beta}\right], \tag{5.22}$$

where Q_a is the charge operator of the gauge group G_a, and $C\left[{\alpha \atop \beta}\right]$ is the internal six-dimensional partition function, which, for the particular case of the $SO(32)$ model, can be read from (4.6). As noticed in [33], only the $\mathcal{N}=2$ sectors of the theory contribute to the moduli dependent part of this expression.

Again, the details of the computation are relegated to Appendix C.3. The expression for the gauge threshold corrections of the heterotic $SO(32)$ model is

$$\begin{aligned} \Lambda_2 = -\frac{1}{96} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \sum_{i=1}^{3} \Big[& (-1)^{m_i+n_i} \hat{Z}_i \bar{\vartheta}_3^2 \bar{\vartheta}_4^2 - \hat{Z}_i^{m_i+\frac{1}{2}, n_i+\frac{1}{2}} \bar{\vartheta}_2^2 \bar{\vartheta}_3^2 \\ & - (-1)^{m_i+n_i} \hat{Z}_i^{m_i+\frac{1}{2}, n_i+\frac{1}{2}} \bar{\vartheta}_2^2 \bar{\vartheta}_4^2 \Big] \frac{\bar{E}_4(\bar{\hat{E}}_2 \bar{E}_4 - \bar{E}_6)}{\bar{\eta}^{24}}, \end{aligned} \tag{5.23}$$

where E_{2n} are the Eisenstein series (given explicitly in Appendix D), and the three toroidal lattice sums, $\hat{Z}_i \equiv |\eta|^4 \Lambda_i$, read

$$\begin{aligned} \hat{\mathbf{Z}}_i\left[{h \atop g}\right] = \frac{\operatorname{Re} T_i}{\tau_2} & \sum_{n_1,\ell^1,n_2,\ell^2} (-1)^{hn_1+g\ell_1} \\ & \times \exp\left[2\pi T_i \det(A) - \frac{\pi(\operatorname{Re} T_i)}{\tau_2(\operatorname{Re} U_i)} \left| (1 \quad iU_i)\, A \begin{pmatrix} \tau \\ 1 \end{pmatrix} \right|^2 \right], \end{aligned} \tag{5.24}$$

with

$$A = \begin{pmatrix} n_1 + \frac{g}{2} & \ell_1 + \frac{h}{2} \\ n_2 & \ell_2 \end{pmatrix} \tag{5.25}$$

and

$$\begin{aligned} & (-1)^{m_i+n_i} \hat{Z}_i = \hat{\mathbf{Z}}_i\left[{1 \atop 0}\right], \qquad \hat{Z}_i^{m_i+\frac{1}{2}, n_i+\frac{1}{2}} = \hat{\mathbf{Z}}_i\left[{0 \atop 1}\right], \\ & (-1)^{m_i+n_i} \hat{Z}_i^{m_i+\frac{1}{2}, n_i+\frac{1}{2}} = \hat{\mathbf{Z}}_i\left[{1 \atop v1}\right]. \end{aligned}$$

Notice that $\left[{h \atop g}\right]_i$ labels the three $\mathcal{N}=2$ sectors associated to the ith 2-torus, $i=1,2,3$. Although the full expression (5.23) is worldsheet modular invariant, each of these $\mathcal{N}=2$ sectors is not worldsheet modular invariant by itself, contrary to what happens in orbifolds with a trivial action on the winding modes.

In the large volume limit, $\operatorname{Re} T_i \gg 1$, the winding modes decouple and only Kaluza–Klein modes with small q contribute to the integral. In that case, the threshold correction receives contributions only from A matrices with zero determinant in the sector $(h,g)=(1,0)$, in such a way that (5.23) becomes[10]

$$\Lambda_2\big|_{\mathrm{Re}\,T_i\gg 1} \simeq \frac{b}{3}\left[-\pi\,\mathrm{Re}\,U_i - \log\left[(\mathrm{Re}\,U_i)(\mathrm{Re}\,T_i)\left|\frac{\eta^3}{\vartheta_4}(2iU_i)\right|^2\mu^2\right]\right], \tag{5.26}$$

matching exactly the threshold corrections for the dual type I $SO(32)$ model.

For arbitrary T_i, however, the winding modes do not decouple from the low energy physics and corrections due to worldsheet instantons appear:

$$\Lambda_2 \simeq \Lambda_2\big|_{\mathrm{Re}\,T_i\gg 1}(U_i) + \Lambda_{\mathrm{inst}}(U_i, T_i). \tag{5.27}$$

They correspond to E1 instanton contributions in the dual type I $SO(32)$ model, and therefore are absent in (5.17).

For example, consider the $q \to 0$ contributions to Λ_{inst} of winding modes in the sector $(h, g) = (1, 0)$. These result in

$$\Lambda_{\mathrm{inst}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\big|_{q\to 0} \simeq -\frac{2b}{3}\sum_{n=1}^{\infty}(-1)^n \log\prod_{i=1}^{3}\left(1-e^{-2\pi n T_i}\right) + \text{c.c.} \tag{5.28}$$

Since the axionic part of T_i in type I corresponds to components of the RR 2-form, C_2, it is natural to expect that these contributions come from E1 instantons wrapping n times the $(1, 1)$-cycle associated to T_i. Notice that the dependence on T_i perfectly agrees with general arguments in [16] for the mirror type IIA picture.

The corresponding holomorphic gauge kinetic function reads

$$f = S - 15\sum_{i=1}^{3}\left[\log\frac{\vartheta_4}{e^{\pi U_i/2}\eta^3}(2iU_i) - 2\sum_{n=1}^{\infty}(-1)^n\log\left(1-e^{-2\pi n T_i}\right)\right] + \cdots, \tag{5.29}$$

where the dots denote further contributions from Λ_{inst}. Hence, the non-perturbative superpotential generated by gaugino condensation receives an extra dependence in the Kähler moduli,

$$W_{np} = A(U_i, T_i)e^{-aS}, \tag{5.30}$$

with

$$a = \frac{1}{15}, \qquad A = \prod_{i=1}^{3}\left[e^{-\pi U_i/2}\frac{\vartheta_4}{\eta^3}(2iU_i)\prod_{n=1}^{\infty}\left(\frac{1-e^{-4\pi(n+1/2)T_i}}{1-e^{-4\pi n T_i}}\right)^2\right]\times\cdots. \tag{5.31}$$

Unfortunately, a complete analytic evaluation of the non-perturbative corrections in (5.23) is subtle, as worldsheet modular invariance mix orbits within different $\mathcal{N} = 2$ sectors and the unfolding techniques of [13,33] cannot be applied straightforwardly to this case.

6. Euclidean brane instantons in the type I freely-acting *SO*(32) model

The model has two types of BPS brane instantons, denoted as E5 and E1. The E5 branes are interpreted as gauge instantons within the four-dimensional gauge theory on the compactified D9 branes and map, in the heterotic dual, to non-perturbative euclidean NS5 corrections. The E1$_i$

[10] We have neglected an extra term coming from the non-holomorphic regularization of $\hat{E}_2$, which in the dual type I side would presumably correspond to contact contributions in two-loop open string diagrams.

Table 2
Op-planes and D9/Ep branes present in the type I models. A – denotes a coordinate parallel to the Op-plane/Dp-brane, while a • represents an orthogonal coordinate

Coord.	0	1	2	3	4	5	6	7	8	9
D9/O9	–	–	–	–	–	–	–	–	–	–
$O5_1$	–	–	–	–	–	–	•	•	•	•
$O5_2$	–	–	–	–	•	•	–	–	•	•
$O5_3$	–	–	–	–	•	•	•	•	–	–
$E1_1$	•	•	•	•	–	–	•	•	•	•
$E1_2$	•	•	•	•	•	•	–	–	•	•
$E1_3$	•	•	•	•	•	•	•	•	–	–
E5	•	•	•	•	–	–	–	–	–	–

type I instantons wrapping the internal torus T^i, instead, are stringy instantons from the gauge theory perspective and are responsible, in the heterotic dual, for the perturbative world-sheet instantons effects, that we have computed in Section 5.[11]

The configurations of the various Op planes and (D/E)p branes in the models are pictorially provided in Table 2.

6.1. E5 instantons

A convenient way to describe the E5 instantons is to write the partition functions coming from the cylinder amplitudes (for E5–E5 and E5–D9 strings) and the Möbius amplitudes (for E5–O9 and E5–O5$_i$). In order to extract the spectrum, it is useful to express the result using the subgroup of $SO(10)$ involved in a covariant description, namely $SO(4)\times SO(2)^3$ in our present case. Considering p coincident E5 instantons, one gets

$$A_{\mathrm{E5-E5}} = \frac{p^2}{16}\int_0^\infty \frac{dt}{t}\frac{1}{\eta^2}\sum_{\alpha\beta} c_{\alpha\beta}\frac{\vartheta\big[{\alpha \atop \beta}\big]}{\eta}\Bigg\{P_1P_2P_3\frac{\vartheta\big[{\alpha \atop \beta}\big]^3}{\eta^9}$$
$$+(-1)^{m_1}P_1\frac{\vartheta\big[{\alpha \atop \beta}\big]\vartheta\big[{\alpha \atop \beta+1/2}\big]\vartheta\big[{\alpha \atop \beta-1/2}\big]}{\eta^5}\frac{4\eta^2}{\vartheta_2^2}$$
$$+(-1)^{m_2}P_2\frac{\vartheta\big[{\alpha \atop \beta-1/2}\big]\vartheta\big[{\alpha \atop \beta}\big]\vartheta\big[{\alpha \atop \beta+1/2}\big]}{\eta^5}\frac{4\eta^2}{\vartheta_2^2}$$
$$+(-1)^{m_3}P_3\frac{\vartheta\big[{\alpha \atop \beta+1/2}\big]\vartheta\big[{\alpha \atop \beta-1/2}\big]\vartheta\big[{\alpha \atop \beta}\big]}{\eta^5}\frac{4\eta^2}{\vartheta_2^2}\Bigg\}, \tag{6.1}$$

$$M_{\mathrm{E5-O9}} = -\frac{p}{16}\int_0^\infty \frac{dt}{t}\frac{4\eta^2}{\vartheta_2^2}\eta^2\sum_{\alpha\beta} c_{\alpha\beta}\frac{\vartheta\big[{\alpha \atop \beta+1/2}\big]\vartheta\big[{\alpha \atop \beta-1/2}\big]}{\eta^2}\frac{\eta}{\vartheta\big[{\alpha \atop \beta}\big]}\Bigg\{P_1P_2P_3\frac{\vartheta\big[{\alpha \atop \beta}\big]^3}{\eta^9}$$

[11] Notice that generically there will be also massless modes stretching between both kind of instantons, E5 and E1$_i$. From the gauge theory perspective, these modes are presumably responsible of the E1 instanton corrections to the Veneziano–Yankielowicz superpotential, discussed at the end of Section 5.3.

$$+\left[(-1)^{m_1} P_1 \frac{\vartheta\left[{\alpha \atop \beta}\right]\vartheta\left[{\alpha \atop \beta+1/2}\right]\vartheta\left[{\alpha \atop \beta-1/2}\right]}{\eta^3}\right.$$
$$+(-1)^{m_2} P_2 \frac{\vartheta\left[{\alpha \atop \beta-1/2}\right]\vartheta\left[{\alpha \atop \beta}\right]\vartheta\left[{\alpha \atop \beta+1/2}\right]}{\eta^3}$$
$$\left.+(-1)^{m_3} P_3 \frac{\vartheta\left[{\alpha \atop \beta+1/2}\right]\vartheta\left[{\alpha \atop \beta-1/2}\right]\vartheta\left[{\alpha \atop \beta}\right]}{\eta^3}\right] \times \frac{1}{\eta^2}\frac{4\eta^2}{\vartheta_2^2}\Bigg\}, \tag{6.2}$$

where $c_{\alpha\beta}$ are the usual GSO projection coefficients. In terms of covariant $SO(4)\times SO(2)^3$ characters, the massless instanton zero-modes content results

$$A^{(0)}_{\text{E5–E5}} + M^{(0)}_{\text{E5–O9}} = \frac{p(p+1)}{2}(V_4 O_2 O_2 O_2 - C_4 C_2 C_2 C_2) - \frac{p(p-1)}{2} S_4 S_2 S_2 S_2. \tag{6.3}$$

From a four-dimensional perspective, $V_4 O_2 O_2 O$ describe vector zero-modes, a_μ, while $C_4 C_2 C_2 C_2$ is a spinor $M^{\alpha,---}$, where α denotes an $SO(4)$ spinor index of positive chirality, whereas $(- - -)$ denote the $SO(2)^3$ internal chiralities. Analogously, $S_4 S_2 S_2 S_2$ are fermionic zero modes $\lambda^{\dot\alpha,---}$. Notice that in the one-instanton $p=1$ sector, λ is projected out by the orientifold projection.

The charged instanton spectrum is obtained from strings stretched between the E5 instanton and the D9 background branes. The corresponding cylinder amplitude is

$$A_{\text{E5–D9}} = \frac{Np}{8}\int_0^\infty \frac{dt}{t}\frac{\eta^2}{\vartheta_4^2}\eta^2 \sum_{\alpha\beta} c_{\alpha\beta} \frac{\vartheta\left[{\alpha+1/2 \atop \beta}\right]^2}{\eta^2}\frac{\eta}{\vartheta\left[{\alpha \atop \beta}\right]}\Bigg\{P_1 P_2 P_3 \frac{\vartheta\left[{\alpha \atop \beta}\right]^3}{\eta^9}$$
$$+\left[(-1)^{m_1} P_1 \frac{\vartheta\left[{\alpha \atop \beta}\right]\vartheta\left[{\alpha \atop \beta+1/2}\right]\vartheta\left[{\alpha \atop \beta-1/2}\right]}{\eta^3}\right.$$
$$+(-1)^{m_2} P_2 \frac{\vartheta\left[{\alpha \atop \beta-1/2}\right]\vartheta\left[{\alpha \atop \beta}\right]\vartheta\left[{\alpha \atop \beta+1/2}\right]}{\eta^3}$$
$$\left.+(-1)^{m_3} P_3 \frac{\vartheta\left[{\alpha \atop \beta+1/2}\right]\vartheta\left[{\alpha \atop \beta-1/2}\right]\vartheta\left[{\alpha \atop \beta}\right]}{\eta^3}\right] \times \frac{1}{\eta^2}\frac{4\eta^2}{\vartheta_2^2}\Bigg\}. \tag{6.4}$$

The massless states are described by the contributions

$$A^{(0)}_{\text{E5–D9}} = Np(S_4 O_2 O_2 O_2 - O_4 C_2 C_2 C_2). \tag{6.5}$$

In particular, the state $S_4 O_2 O_2 O_2$, coming from the NS sector, has a spinorial $SO(4)$ index ω_α, whereas $O_4 C_2 C_2 C_2$, coming from the R sector, is an $SO(4)$ scalar with a spinorial $SO(6)$ index or, which is the same, a fundamental $SU(4)$ index μ^A.

6.2. E1 instantons

The case of the E1 instantons is more subtle. Indeed, they wrap one internal torus while they are orthogonal to the two remaining ones, thus feeling the non-trivial effects of the freely-acting operations. The explicit discussion can be limited to the case of the E1_1 instantons, the other two cases $\text{E1}_{2,3}$ being obviously completely similar. It is useful to separately discuss the two distinct possibilities:

(i) the $E1_1$ instantons sit at one of the fixed points (tori) of the g orbifold generator in the $y^1, \dots, y^6$ directions;
(ii) the $E1_1$ instantons are located off the fixed points (tori) of the g orbifold generator in the $y^1, \dots, y^6$ directions.

It is worth to stress that, strictly speaking, the freely action g has no fixed tori, due, of course, to the shift along T^1. However, since the instanton $E1_1$ wraps T^1, while it is localized in the (T^2, T^3) directions, it is convenient to analyze the orbifold action in the space perpendicular to the instanton world-volume.

In the following, we discuss the first configurations with the instantons on the fixed tori, which are the relevant ones for matching the dual heterotic threshold corrections. Since the freely-acting operations (f, h) identify points in the internal space perpendicular to the instanton world-volume, they enforce the presence of doublets of $E1_1$ instantons, in complete analogy with similar phenomena happening in the case of background D5 branes in [18,19]. Indeed, the g-operation is the only one acting in a non-trivial way on the instantons. The doublet nature of the $E1_1$ instantons can be explicitly figured out in the following geometric way. Let the location of the $E1_1$ instanton be fixed at a point of the (y^3, y^4, y^5, y^6) space, which is left invariant by the g-operation. For instance, $|E1_1\rangle = |0, 0, \pi R_5/2, 0\rangle$. Then, the f and h operations both map the point $|E1_1\rangle$ into $|E1_1'\rangle = |\pi R_3, 0, 3\pi R_5/2, 0\rangle$, so that an orbifold invariant instanton state is provided by the combination ("doublet")

$$\frac{1}{\sqrt{2}}\big[|0, 0, \pi R_5/2, 0\rangle + |\pi R_3, 0, 3\pi R_5/2, 0\rangle\big]. \tag{6.6}$$

The corresponding open strings can be stretched between fixed points and/or images, and can be described by the following amplitudes

$$\begin{aligned} A_{\mathrm{E1-E1}} = \frac{q^2}{32}\int_0^\infty \frac{dt}{t}\frac{1}{\eta^2}\sum_{\alpha\beta} c_{\alpha\beta}\frac{\vartheta\big[{\alpha \atop \beta}\big]}{\eta}\bigg\{ & P_1\big(W_2W_3 + W_2^{n+1/2}W_3^{n+1/2}\big)\frac{\vartheta\big[{\alpha \atop \beta}\big]^3}{\eta^9} \\ & + (-1)^{m_1}P_1\frac{\vartheta\big[{\alpha \atop \beta}\big]\vartheta\big[{\alpha \atop \beta+1/2}\big]\vartheta\big[{\alpha \atop \beta-1/2}\big]}{\eta^5}\frac{4\eta^2}{\vartheta_2^2}\bigg\}, \end{aligned} \tag{6.7}$$

$$\begin{aligned} M_{\mathrm{E1-O9}} = -\frac{q}{16}\int_0^\infty \frac{dt}{t}\frac{4\eta^2}{\vartheta_2^2}\eta^2\sum_{\alpha\beta} c_{\alpha\beta}\frac{\vartheta\big[{\alpha \atop \beta+1/2}\big]\vartheta\big[{\alpha \atop \beta-1/2}\big]}{\eta^2}\frac{\eta}{\vartheta\big[{\alpha \atop \beta}\big]}\bigg\{ & (-1)^{m_1}P_1W_2W_3\frac{\vartheta\big[{\alpha \atop \beta}\big]^3}{\eta^9} \\ & + P_1\frac{\vartheta\big[{\alpha \atop \beta}\big]\vartheta\big[{\alpha \atop \beta+1/2}\big]\vartheta\big[{\alpha \atop \beta-1/2}\big]}{\eta^3}\frac{1}{\eta^2}\frac{4\eta^2}{\vartheta_2^2}\bigg\}. \end{aligned} \tag{6.8}$$

Since only the Z_2 g-operation acts non-trivially on the characters, it is convenient in this case to use covariant $SO(4) \times SO(2) \times SO(4)$ characters in order to describe the massless instanton zero-modes. Due to the doublet nature of the instantons, particle interpretation asks for a rescaling of the "charge" $q = 2Q$, meaning that the tension of the elementary instanton is twice the tension of the standard D1-brane. The result is

$$\begin{aligned} A^{(0)}_{\mathrm{E1-E1}} + M^{(0)}_{\mathrm{E1-O9}} = {} & \frac{Q(Q+1)}{2}(V_4O_2O_4 - C_4C_2S_4) \\ & + \frac{Q(Q-1)}{2}(O_4V_2O_4 - S_4S_2S_4). \end{aligned} \tag{6.9}$$

These zero-modes describe the positions x^μ of the E1 instantons in spacetime, scalars y^i along the torus wrapped by the instanton and fermions $\Theta^{\dot\alpha,-,a}$, $\Theta^{\alpha,+,a}$. The charged $\mathrm{E1}_1$–D9 instanton spectrum is obtained from strings stretched between the E1 instantons and the D9 background branes. The corresponding cylinder amplitude is

$$A_{\mathrm{E1-D9}} = \frac{Nq}{8}\int_0^\infty \frac{dt}{t}\frac{\eta^2}{\vartheta_4^2}\eta^2\sum_{\alpha\beta}c_{\alpha\beta}\frac{\vartheta\left[{\alpha+1/2 \atop \beta}\right]^2}{\eta^2}\frac{\eta}{\vartheta\left[{\alpha \atop \beta}\right]}\left\{P_1\frac{\vartheta\left[{\alpha \atop \beta}\right]\vartheta\left[{\alpha+1/2 \atop \beta}\right]^2}{\eta^3}\frac{\eta^2}{\vartheta_4^2}\right.$$
$$\left. + (-1)^{m_1}P_1\frac{\vartheta\left[{\alpha \atop \beta}\right]\vartheta\left[{\alpha+1/2 \atop \beta+1/2}\right]\vartheta\left[{\alpha+1/2 \atop \beta-1/2}\right]}{\eta^3}\frac{\eta^2}{\vartheta_3^2}\right\}. \tag{6.10}$$

The surviving massless states are now described by

$$A^{(0)}_{\mathrm{E1-D9}} = NQ(-O_4S_2O_4), \tag{6.11}$$

and correspond to the surviving "would be" world-sheet current algebra fermionic modes in the "heterotic string" interpretation (with $Q=1$ and $N=32$ [11,35]).

The second configuration, where the $\mathrm{E1}_1$ instantons are off the fixed points (tori) of the g orbifold generator in $y^1\dots y^6$, for instance $|\mathrm{E1}_1\rangle = |0,0,0,0\rangle$, can be worked out as well. In this case a quartet structure of instantons is present, in a situation again similar to the ones described in [18,19]. Indeed, g produces the image $g:|0,0,0,0\rangle \to |0,0,\pi R_5,0\rangle$, while f and h produce two other images $f:|0,0,0,0\rangle \to |\pi R_3,0,0,0\rangle$, $h:|0,0,0,0\rangle \to |\pi R_3,0,\pi R_5,0\rangle$. In conclusion, the orbifold-invariant linear superposition of the instanton images is now the combination

$$\frac{1}{2}\big[|0,0,0,0\rangle + |0,0,\pi R_5,0\rangle + |\pi R_3,0,0,0\rangle + |\pi R_3,0,\pi R_5,0\rangle\big]. \tag{6.12}$$

For a given number of "bulk" E1 instantons, they have twice the number of neutral (uncharged) fermionic zero modes as compared to their "fractional" instantons cousins (6.9), whose minimal number of uncharged zero modes is four. On the other hand, their tension is twice bigger. If n "fractional" E1 instanton doublets wrap the torus T^i, one expects a contribution proportional to $e^{-4\pi n T_i}$, whereas if they wrap half of the internal torus, consistently with the shift identification, the contributions should be proportional to $e^{-4\pi(n+1/2)T_i}$. These considerations are perfectly in agreement with the $\mathcal{N}=2$ nature of the threshold corrections appearing in the heterotic computation (5.23), (5.29) and (5.31). On the other hand, the quartet structure of the "bulk" instantons is probably incompatible with them. It should be also noticed that the absence of $\mathcal{N}=1$ sectors contributing to the threshold corrections (moduli-independent threshold corrections) on the heterotic side reflects the fact that only the f and h action create instanton images.

A similar analysis to the one carried out in this section can be performed for the more general type I $SO(n_o)\otimes SO(n_g)\otimes SO(n_f)\otimes SO(n_h)$ model presented in Section 3.1. However, we do not find any remarkable difference in nature between different choices of n_o, n_g, n_f and n_h, contrary to what the heterotic dual model seems to suggest. It would be interesting to clarify this issue and to understand why type I models differing only in the Chan–Paton charges lead to so different models in the heterotic dual side.

7. Fluxes and moduli stabilization

7.1. $\mathbb{Z}_2 \times \mathbb{Z}_2$ freely-acting orbifolds of twisted tori

Background fluxes for the RR and NSNS fields have been shown to be relevant for lifting some of the flat directions of the closed string moduli space. From the four-dimensional effective field theory perspective, the lift can be properly understood in terms of a non-trivial superpotential encoding the topological properties of the background. Many models based on ordinary Abelian orientifolds of string theory have appeared in the literature (for recent reviews and references see for instance [36]). Here we would like to extend this construction to the case of orientifolds with a free action. The motivation is two-fold. First, in these models the twisted sector modes are massive, as has been previously shown. The same happens for the open string moduli transforming in the adjoint. Second, we have enough control over the non-perturbative regime, so that this model provides us with a laboratory on which to explicitly test the combined effect of fluxes and non-perturbative effects.

For the particular type I (heterotic) orbifolds considered here, the orientifold projection kills a possible constant H_3 (F_3) background, so that the only possibilities left, apart from non-geometric deformations, are RR (NSNS) 3-form fluxes and metric fluxes [37–39]. The latter correspond to twists of the cohomology of the internal manifold $\mathcal{M}$,

$$d\omega_i = M_i{}^j \alpha_j + N^i{}_j \beta^j, \tag{7.1}$$

where ω_i is a basis of harmonic 2-forms in $\mathcal{M}$, and (α_i, β^j) a symplectic basis of harmonic 3-forms. The resulting manifold $\tilde{\mathcal{M}}$ is in general no longer Calabi–Yau, but rather it possesses $SU(3)$-structure [38,40]. Duality arguments show, however, that the light modes of the compactification in $\tilde{\mathcal{M}}$ can be suitably described in terms of a compactification in $\mathcal{M}$, together with a non-trivial superpotential W_{twist} accounting for the different moduli spaces.

Here we want to take a further step in the models of the previous sections and to consider geometries which go beyond the toroidal one by adding metric fluxes to the original torus. In terms of the global 1-forms of the torus, the cohomology twist reads,

$$de^i = \frac{1}{2} f^i_{jk} e^j \wedge e^k, \tag{7.2}$$

the resulting manifold being a group manifold $\tilde{\mathcal{M}} = G/\Gamma$ with structure constants f^i_{jk} and Γ a discrete subgroup of G. Modding (7.2) by the orbifold action (2.8)–(2.10) will in general put restrictions on the structure constants f^i_{jk} and the lattice Γ. More concretely, the surviving structure constants are

$$\begin{pmatrix} f^2_{35} \\ f^4_{51} \\ f^6_{14} \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}, \qquad \begin{pmatrix} -f^1_{35} & f^4_{52} & f^6_{23} \\ f^2_{45} & -f^3_{51} & f^6_{14} \\ f^2_{36} & f^4_{61} & -f^5_{13} \end{pmatrix} = -\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

$$\begin{pmatrix} f^1_{46} \\ f^3_{62} \\ f^5_{24} \end{pmatrix} = \begin{pmatrix} \bar{h}_1 \\ \bar{h}_2 \\ \bar{h}_3 \end{pmatrix}, \qquad \begin{pmatrix} -f^2_{46} & f^3_{61} & f^5_{14} \\ f^1_{36} & -f^4_{62} & f^5_{23} \\ f^1_{45} & f^3_{52} & -f^6_{24} \end{pmatrix} = -\begin{pmatrix} \bar{b}_{11} & \bar{b}_{12} & \bar{b}_{13} \\ \bar{b}_{21} & \bar{b}_{22} & \bar{b}_{23} \\ \bar{b}_{31} & \bar{b}_{32} & \bar{b}_{33} \end{pmatrix},$$

as in an ordinary $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. The Jacobi identity of the algebra G requires in addition $f^i_{[jk} f^m_{o]i} = 0$ [22,37]. The set of metric fluxes transforms trivially under S-duality, so one can build heterotic-type I dual pairs by simply exchanging $F_3 \leftrightarrow H_3$.

The low energy physics of the $G/[\Omega_P \times \Gamma \times (\tilde{\mathbb{Z}}_2 \times \tilde{\mathbb{Z}}_2)]$ compactification can be then suitably described in terms of a $T^6/[\Omega_P \times (\mathbb{Z}_2 \times \mathbb{Z}_2)]$ compactification, with $\mathbb{Z}_2 \times \mathbb{Z}_2$ being the freely-acting orbifold action described in Section 2, together with a superpotential [41],

$$W_{\text{twist}} = \sum_{i=1}^{3} T_i \Bigg[-i\bar{h}_i + \sum_{j=1}^{3} \bar{b}_{ji} U_j + i b_{1i} U_2 U_3 + b_{2i} U_1 U_3 + i b_{3i} U_1 U_2 - h_i U_1 U_2 U_3 \Bigg]. \tag{7.3}$$

Notice that the freely-acting $\tilde{\mathbb{Z}}_2 \times \tilde{\mathbb{Z}}_2$ orbifold of the full ten-dimensional picture will in general differ from the freely-acting $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of the effective description. For illustration, consider the following simple example given by,

$$de^1 = b_{11} e^3 \wedge e^5, \qquad de^2 = de^3 = de^4 = de^5 = de^6 = 0. \tag{7.4}$$

We may integrate these equations as,

$$e^1 = dy^1 + b_{11} y^3\, dy^5, \qquad e^i = dy^i \quad \text{for } i \neq 1, \tag{7.5}$$

so that G is a fibration of y^5 over y^1. The lattice Γ is then suitably chosen as,

$$\Gamma: \quad \begin{cases} y^3 \to y^3 + 1, & y^1 \to y^1 - b_{11} y^5, \\ y^i \to y^i + 1 & \text{for } i \neq 3, \end{cases} \tag{7.6}$$

with $b_{11} \in \mathbb{Z}$ so that the vielbein vectors remain invariant under Γ transformations. Acting now with the orbifold generators (2.8)–(2.10), it is not difficult to convince oneself that in order the vielbein vectors to transform covariantly, the orbifold generators have to be replaced by some new ones $\{\tilde{g}, \tilde{f}, \tilde{h}\}$ defined as,

$$\begin{aligned}
&(y^1, y^2, y^3, y^4, y^5, y^6) \xrightarrow{\tilde{g}} (y^1 + 1/2, y^2, -y^3, -y^4, -y^5 + 1/2, -y^6),\\
&(y^1, y^2, y^3, y^4, y^5, y^6) \xrightarrow{\tilde{f}} (-y^1 + 1/2 + b_{11} y^5/2, -y^2, y^3 + 1/2, y^4, -y^5, -y^6),\\
&(y^1, y^2, y^3, y^4, y^5, y^6) \xrightarrow{\tilde{h}} (-y^1 - b_{11} y^5/2, -y^2, -y^3 + 1/2, -y^4, y^5 + 1/2, y^6).
\end{aligned} \tag{7.7}$$

The generators $\{\tilde{g}, \tilde{f}, \tilde{h}\}$ still define a $\mathbb{Z}_2 \times \mathbb{Z}_2$ discrete group. Indeed, requiring the quantization condition $b_{11} \in 2\mathbb{Z}$, one can prove that $\tilde{g}^2 = \tilde{h}^2 = \tilde{f}^2 = 1$ and $\tilde{g}\tilde{f} = \tilde{f}\tilde{g} = \tilde{h}$, $\tilde{g}\tilde{h} = \tilde{h}\tilde{g} = \tilde{f}$, $\tilde{h}\tilde{f} = \tilde{f}\tilde{h} = \tilde{g}$, up to discrete transformations of the lattice Γ. Hence, the light modes of the $SU(3)$-structure orientifold defined by the group manifold (7.5), together with the lattice (7.6) and the orbifold generators (7.7), can be consistently described by a T^6 compactification with an orbifold action given by Eqs. (2.8) and a superpotential term,

$$W_{\text{twist}} = i b_{11} T_1 U_2 U_3. \tag{7.8}$$

7.2. *Moduli stabilization in an $S^3 \times T^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ orbifold*

To illustrate the interplay between non-perturbative effects and metric fluxes we consider in this section the following one-parameter family of twists,

$$de^1 = \alpha e^4 \wedge e^6, \qquad de^2 = \alpha e^4 \wedge e^6,$$

$$de^3 = \alpha e^6 \wedge e^2, \qquad de^4 = \alpha e^6 \wedge e^2,$$
$$de^5 = \alpha e^2 \wedge e^4, \qquad de^6 = \alpha e^2 \wedge e^4.$$

The particular solution to these equations

$$e^1 = dy^1 + e^2, \quad e^2 = \sin(\alpha y^6)\, dy^4 + \cos(\alpha y^6)\cos(\alpha y^4)\, dy^2,$$
$$e^3 = dy^3 + e^4, \quad e^4 = -\cos(\alpha y^6)\, dy^4 + \sin(\alpha y^6)\cos(\alpha y^4)\, dy^2,$$
$$e^5 = dy^5 + e^6, \quad e^6 = dy^6 + \sin(\alpha y^4)\, dy^2,$$

is corresponding to a product of a 3-sphere and a 3-torus. Consistency requires α to be multiple of 2π. On the other hand, in this particular case the orbifold action remains unaffected by the fluxes and is still given by (2.8)–(2.10).

We will also add a possible RR 3-form flux along the 3-sphere,

$$F_3 = me^2 \wedge e^4 \wedge e^6. \tag{7.9}$$

One may easily check that this flux, together with the above twists, does not give rise to tadpole contributions.

The model can be effectively described by a $T^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ compactification with Kähler potential and superpotential,

$$K = -\log(S + S^*) - \sum_{i=1}^{3} \log(U_i + U_i^*) - \sum_{i=1}^{3} \log(T_i + T_i^*), \tag{7.10}$$

$$W = m + \alpha \sum_{j=1}^{3} T_j(-i + U_j) + W_{np}(S, T_1, T_2, T_3, U_1, U_2, U_3), \tag{7.11}$$

where we have introduced a generic non-perturbative superpotential possibly depending on all moduli, as shown in the previous sections.[12]

For $\mathrm{Re}\, T_i \gg 1$ and $\mathrm{Re}\, U_i \gg 1$, the dependence of the non-perturbative superpotential on the Kähler and complex structure moduli can be neglected, $\partial_{U_i} W_{np} \simeq \partial_{T_i} W_{np} \simeq 0$, and the above superpotential has a perturbative vacuum given by

$$\begin{aligned} &\mathrm{Im}\, U_i \simeq 1, \qquad \mathrm{Re}\, W_{np} + m \simeq \alpha(\mathrm{Re}\, T_i)(\mathrm{Re}\, U_i), \\ &\mathrm{Im}\, T_i \simeq 0, \qquad \mathrm{Im}\, W_{np} \simeq 0, \qquad D_S W = 0, \end{aligned} \tag{7.12}$$

with $D_S W = \partial_S W - (S + S^*)^{-1} W$, as usual. Then, for W_{np} the racetrack superpotential (5.20), one may stabilize S at a reasonably not too big coupling.

The model can be viewed in the S-dual heterotic side as an asymmetric $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of some Freedman–Gibbons electrovac solution [43,44].[13] In particular, the full string ground state includes an $SU(2)$ Wess–Zumino–Witten model describing the radial stabilization of the 3-sphere by m units of H_3 flux, provided by $F_3 \to H_3$ in (7.9). In terms of the radii R_i, $i = 1, \ldots, 6$, Eqs. (7.12) lead to

$$(R_2)^2 = (R_4)^2 = (R_6)^2 \simeq \frac{\mathrm{Re}\, W_{np} + m}{\alpha}, \tag{7.13}$$

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[12] Perturbative corrections to the Kähler potential could also play a role in the moduli stabilization. We restrict here to the tree-level form of the Kähler potential, for the possible effect of α' or quantum corrections to it, see, e.g., [42].

[13] We thank E. Kiritsis for pointing out to us this connection.

whereas the radii of the 3-torus, R_1, R_3, R_5, remain as flat directions. Having $\mathrm{Re}\, T_i \gg 1$ and $\mathrm{Re}\, U_i \gg 1$ then requires the volume of the 3-sphere to be much bigger than the volume of the 3-torus, i.e., $m/\alpha \gg 1$.

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Appendix A. Normalization of string amplitudes

For sake of brevity, throughout the paper we ignored the overall factors coming from integrating over the non-compact momenta. For arbitrary string tension α', the complete string amplitudes $\mathcal{T}, \mathcal{K}, \mathcal{A}, \mathcal{M}$ are related to the ones used in the main text by

$$\begin{aligned} \mathcal{T} &= \frac{1}{(4\pi^2\alpha')^2} T, \qquad \mathcal{K} = \frac{1}{(8\pi^2\alpha')^2} K, \\ \mathcal{A} &= \frac{1}{(8\pi^2\alpha')^2} A, \qquad \mathcal{M} = \frac{1}{(8\pi^2\alpha')^2} M. \end{aligned} \tag{A.1}$$

Appendix B. Characters for $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds

In the light-cone RNS formalism, the vacuum amplitudes involve the following characters

$$\begin{aligned}
\tau_{oo} &= V_2I_2I_2I_2 + I_2V_2V_2V_2 - S_2S_2S_2S_2 - C_2C_2C_2C_2, \\
\tau_{og} &= I_2V_2I_2I_2 + V_2I_2V_2V_2 - C_2C_2S_2S_2 - S_2S_2C_2C_2, \\
\tau_{oh} &= I_2I_2I_2V_2 + V_2V_2V_2I_2 - C_2S_2S_2C_2 - S_2C_2C_2S_2, \\
\tau_{of} &= I_2I_2V_2I_2 + V_2V_2I_2V_2 - C_2S_2C_2S_2 - S_2C_2S_2C_2, \\
\tau_{go} &= V_2I_2S_2C_2 + I_2V_2C_2S_2 - S_2S_2V_2I_2 - C_2C_2I_2V_2, \\
\tau_{gg} &= I_2V_2S_2C_2 + V_2I_2C_2S_2 - S_2S_2I_2V_2 - C_2C_2V_2I_2, \\
\tau_{gh} &= I_2I_2S_2S_2 + V_2V_2C_2C_2 - C_2S_2V_2V_2 - S_2C_2I_2I_2, \\
\tau_{gf} &= I_2I_2C_2C_2 + V_2V_2S_2S_2 - S_2C_2V_2V_2 - C_2S_2I_2I_2, \\
\tau_{ho} &= V_2S_2C_2I_2 + I_2C_2S_2V_2 - C_2I_2V_2C_2 - S_2V_2I_2S_2, \\
\tau_{hg} &= I_2C_2C_2I_2 + V_2S_2S_2V_2 - C_2I_2I_2S_2 - S_2V_2V_2C_2, \\
\tau_{hh} &= I_2S_2C_2V_2 + V_2C_2S_2I_2 - S_2I_2V_2S_2 - C_2V_2I_2C_2, \\
\tau_{hf} &= I_2S_2S_2I_2 + V_2C_2C_2V_2 - C_2V_2V_2S_2 - S_2I_2I_2C_2, \\
\tau_{fo} &= V_2S_2I_2C_2 + I_2C_2V_2S_2 - S_2V_2S_2I_2 - C_2I_2C_2V_2,
\end{aligned}$$

$$\tau_{fg} = I_2C_2I_2C_2 + V_2S_2V_2S_2 - C_2I_2S_2I_2 - S_2V_2C_2V_2,$$
$$\tau_{fh} = I_2S_2I_2S_2 + V_2C_2V_2C_2 - C_2V_2S_2V_2 - S_2I_2C_2I_2,$$
$$\tau_{ff} = I_2S_2V_2C_2 + V_2C_2I_2S_2 - C_2V_2C_2I_2 - S_2I_2S_2V_2, \tag{B.1}$$

where each term is a tensor product of the characters of the vector representation (V_2), the scalar representation (I_2), the spinor representation (S_2) and the conjugate-spinor representation (C_2) of the four $SO(2)$ factors that enter the light-cone restriction of the ten-dimensional Lorentz algebra.

Appendix C. Details on the threshold correction computations

C.1. Threshold corrections in the type I $SO(n_o) \otimes SO(n_g) \otimes SO(n_f) \otimes SO(n_h)$ models

In order to implement the background field method, it is convenient to express the orbifold characters in terms of the corresponding ϑ-functions:

$$\tau_{oo} + \tau_{og} + \tau_{oh} + \tau_{of} = \frac{1}{2\eta^4}\big(\vartheta_3^4 - \vartheta_4^4 - \vartheta_2^4 - \vartheta_1^4\big), \tag{C.1}$$

$$\tau_{oo} + \tau_{og} - \tau_{oh} - \tau_{of} = \frac{1}{2\eta^4}\big(\vartheta_2^2\vartheta_1^2 + \vartheta_1^2\vartheta_2^2 - \vartheta_4^2\vartheta_3^2 + \vartheta_3^2\vartheta_4^2\big), \tag{C.2}$$

$$\tau_{oo} - \tau_{og} + \tau_{oh} - \tau_{of} = \frac{1}{2\eta^4}\big(\vartheta_1\vartheta_2^2\vartheta_1 + \vartheta_2\vartheta_1^2\vartheta_2 + \vartheta_3\vartheta_4^2\vartheta_3 - \vartheta_4\vartheta_3^2\vartheta_4\big), \tag{C.3}$$

$$\tau_{oo} - \tau_{og} - \tau_{oh} + \tau_{of} = \frac{1}{2\eta^4}(\vartheta_2\vartheta_1\vartheta_2\vartheta_1 + \vartheta_1\vartheta_2\vartheta_1\vartheta_2 - \vartheta_4\vartheta_3\vartheta_4\vartheta_3 + \vartheta_3\vartheta_4\vartheta_3\vartheta_4). \tag{C.4}$$

Making use of the expansion (valid for even spin structure α)

$$\frac{\vartheta_\alpha(\epsilon\tau|\tau)}{\vartheta_1(\epsilon\tau|\tau)} = \frac{1}{2\pi\epsilon\tau}\frac{\vartheta_\alpha}{\eta^3} + \frac{\epsilon\tau}{4\pi}\frac{\vartheta''_\alpha}{\eta^3} + \cdots, \tag{C.5}$$

and the modular identities (D.2) and (D.3) in Appendix D, the expansions of the characters in terms of the (small) magnetic field or, equivalently, in terms of the ϵ of Eq. (5.8), are

$$\begin{aligned}
(\tau_{oo} + \tau_{og} + \tau_{oh} + \tau_{of})(\epsilon\tau,\tau) &\simeq -\frac{i\epsilon\tau}{8\pi\eta^4}\big(\vartheta_3''\vartheta_3^3 - \vartheta_4''\vartheta_4^3 - \vartheta_2''\vartheta_2^3\big) = 0,\\
(\tau_{oo} + \tau_{og} - \tau_{oh} - \tau_{of})(\epsilon\tau,\tau) &= (\tau_{oo} - \tau_{og} - \tau_{oh} + \tau_{of})(\epsilon\tau,\tau)\\
&= (\tau_{oo} - \tau_{og} + \tau_{oh} - \tau_{of})(\epsilon\tau,\tau)\\
&\simeq -\frac{i\epsilon\tau}{8\pi\eta^4}\big(-\vartheta_4''\vartheta_4\vartheta_3^2 + \vartheta_3''\vartheta_3\vartheta_4^2\big) = \frac{i\pi\epsilon}{2}\tau\eta^2\vartheta_2^2.
\end{aligned} \tag{C.6}$$

The one-loop threshold corrections on any of the gauge group factors can therefore be written in the form

$$\begin{aligned}
\Lambda_2 = 16\pi^2\int_0^\infty \frac{dt}{t}\big\{&\big[2\,\mathrm{Tr}\big(Q^2\big) - \mathrm{Tr}(\gamma_g)\,\mathrm{Tr}\big(\gamma_g Q^2\big)\big](-1)^{m_1}P_1\\
&+ \big[2\,\mathrm{Tr}\big(Q^2\big) - \mathrm{Tr}(\gamma_f)\,\mathrm{Tr}\big(\gamma_f Q^2\big)\big](-1)^{m_2}P_2\\
&+ \big[2\,\mathrm{Tr}\big(Q^2\big) - \mathrm{Tr}(\gamma_h)\,\mathrm{Tr}\big(\gamma_h Q^2\big)\big](-1)^{m_3}P_3\big\},
\end{aligned} \tag{C.7}$$

where the action induced by the orbifold on the CP matrices, defined in (3.10), has been used. The last step is to compute the momentum sums $(-1)^m P$. To this end, it is useful to reexpress (3.7) as

$$\begin{aligned}\Gamma &\equiv \int_0^\infty \frac{dt}{t}(-1)^m P\\ &= \int_0^\infty \frac{dt}{t}\exp\left[-\frac{\pi(\operatorname{Re}T)}{4t(\operatorname{Re}U)}\right]\sum_{m,m'}\exp\left[-\pi(m-b)^T A(m-b)\right],\end{aligned}\tag{C.8}$$

with

$$m-b=\begin{pmatrix} m-\frac{i(\operatorname{Re}T)}{2t(\operatorname{Re}U)}\\ m'+\frac{i(\operatorname{Re}T)(\operatorname{Im}U)}{2t(\operatorname{Re}U)}\end{pmatrix},\qquad A=\frac{t}{(\operatorname{Re}T)(\operatorname{Re}U)}\begin{pmatrix}|U|^2 & \operatorname{Im}U\\ \operatorname{Im}U & 1\end{pmatrix}.\tag{C.9}$$

Making use of the Poisson summation formula (D.1) and redefining $t\to 1/\ell$ in order to move to the transverse channel picture, one gets

$$\Gamma=(\operatorname{Re}T)\int_0^\infty d\ell\sum_{n_1,n_2}\exp\left[-\frac{\pi\ell(\operatorname{Re}T)}{\operatorname{Re}U}\left[\left(n_1+\frac{1}{2}-n_2\operatorname{Im}U\right)^2+(n_2\operatorname{Re}U)^2\right]\right].\tag{C.10}$$

As expected, the integral contains infrared (IR) divergences as $\ell\to 0$, corresponding to loops of massless modes. It can be regularized introducing an IR regulator μ via a factor $F_\mu=(1-e^{-l/\mu^2})$. Performing the integral in ℓ the result is

$$\begin{aligned}\Gamma=\lim_{\mu^2\to 0}\Bigg[\frac{\operatorname{Re}U}{\pi}\sum_{n_1,n_2}\Bigg(&\frac{1}{(n_1+\frac{1}{2}-n_2\operatorname{Im}U)^2+(n_2\operatorname{Re}U)^2}\\ &-\frac{1}{(n_1+\frac{1}{2}-n_2\operatorname{Im}U)^2+(n_2\operatorname{Re}U)^2+\frac{\operatorname{Re}U}{\pi\mu^2\operatorname{Re}T}}\Bigg)\Bigg].\end{aligned}\tag{C.11}$$

Finally, using the Dixon, Kaplunovsky and Louis (DKL) formula [33] to evaluate the sum over n_1, the expression become

$$\Gamma=-\sum_{n_2>0}\left[\frac{1}{n_2}\left(\frac{q^{n_2}-1}{q^{n_2}+1}+\frac{\bar q^{n_2}-1}{\bar q^{n_2}+1}\right)+\frac{2}{\sqrt{n_2^2+(1/\pi(\operatorname{Re}U)(\operatorname{Re}T)\mu^2)}}\right],\tag{C.12}$$

with $q\equiv\exp[-2\pi U]$ and where we have taken $\mu^2\ll 1$ (in string units). A Taylor expansion (using Eq. (D.19)) produces

$$\begin{aligned}\Gamma&=\sum_{n_2>0}\left(\frac{2}{n_2}-\frac{2}{\sqrt{n_2^2+(1/\pi(\operatorname{Re}U)(\operatorname{Re}T)\mu^2)}}\right)\\ &\quad+2\sum_{n_2,m>0}\frac{(-1)^m}{n_2}q^{mn_2}+2\sum_{n_2,m>0}\frac{(-1)^m}{n_2}\bar q^{mn_2}\\ &=\sum_{n_2>0}\left(\frac{2}{n_2}-\frac{2}{\sqrt{n_2^2+(1/\pi(\operatorname{Re}U)(\operatorname{Re}T)\mu^2)}}\right)\end{aligned}$$

$$-2\sum_{m>0}\log\bigl(1-q^{2m}\bigr)+2\sum_{m>0}\log\bigl(1-q^{2m-1}\bigr)+\text{c.c.} \tag{C.13}$$

Taking the $\mu^2 \to 0$ limit and at the same time subtracting the finite[14] and the cut-off dependent parts, in terms of the modular functions (D.17) and (D.16) one gets

$$\int_0^\infty \frac{dt}{t}(-1)^m F_\mu P = \log\left|\frac{\vartheta_4}{\eta^3}(2iU)\right|^2 - \pi\,\mathrm{Re}\,U - \log\bigl[(\mathrm{Re}\,U)(\mathrm{Re}\,T)\mu^2\bigr]. \tag{C.14}$$

C.2. Threshold corrections in the type I racetrack models

The procedure for the racetrack models is completely analogous to the one in the previous section. Plugging (C.6) into (3.11) and (3.12) one gets

$$\begin{aligned}\Lambda_{2,p} = 16\pi^2\,\mathrm{Tr}\bigl(Q^2\bigr)\int_0^\infty \frac{dt}{t}\bigl[\bigl[(2-p)P_1 - q\bigl(P_{m'+\frac{1}{2}}P_{m_1}\bigr)\bigr](-1)^{m_1}\\ + (2-p)P_2(-1)^{m_2} + (2-p)P_3(-1)^{m_3}\bigr],\end{aligned} \tag{C.15}$$

where the Q generator has been taken in the $SO(p)$ factor. In this case there is a new lattice summation to compute, namely

$$\begin{aligned}\Gamma' &= \int_0^\infty \frac{dt}{t}(-1)^m P_{m'+\frac{1}{2}}P_m\\ &= \int_0^\infty \frac{dt}{t}\sum_{m,m'}(-1)^m \exp\left[-\frac{\pi t}{(\mathrm{Re}\,T)(\mathrm{Re}\,U)}\left|m'+\frac{1}{2}-iUm\right|^2\right]\\ &= \int_0^\infty \frac{dt}{t}\exp\left[-\frac{\pi(\mathrm{Re}\,T)}{4t(\mathrm{Re}\,U)}\right]\sum_{m,m'}\exp\bigl[-\pi(m-b)^T A(m-b)\bigr],\end{aligned} \tag{C.16}$$

where now

$$m-b = \begin{pmatrix} m - \frac{i(\mathrm{Re}\,T)}{2t(\mathrm{Re}\,U)} \\ m' + \frac{i(\mathrm{Re}\,T)(\mathrm{Im}\,U)}{2t(\mathrm{Re}\,U)} + \frac{1}{2}\end{pmatrix}, \qquad A = \frac{t}{(\mathrm{Re}\,T)(\mathrm{Re}\,U)}\begin{pmatrix} |U|^2 & \mathrm{Im}\,U \\ \mathrm{Im}\,U & 1\end{pmatrix}. \tag{C.17}$$

Thus, the integration in the transverse channel gives

$$\Gamma' = \frac{\mathrm{Re}\,U}{\pi}\sum_{n_1,n_2}\frac{(-1)^{n_2}}{(n_1+\frac{1}{2}-n_2\,\mathrm{Im}\,U)^2+(n_2\,\mathrm{Re}\,U)^2}. \tag{C.18}$$

Using again the (DKL) formula, after some algebra, the Γ' can be written

$$\Gamma' = \sum_{n_2>0}\frac{1}{n_2}\left(\frac{q^{n_2}-1}{q^{n_2}+1} - \frac{q^{2n_2}-1}{q^{2n_2}+1}\right) + \text{c.c.}, \tag{C.19}$$

[14] The finite term can be actually reabsorbed into the value of the gauge coupling at the compactification scale.

with $q = \exp[-2\pi U]$. It should be noticed that in this case there is no need of an IR regulator for this sum. In terms of modular functions the integral becomes

$$\int_0^\infty \frac{dt}{t}(-1)^m P_{m'+\frac{1}{2}} P_m = \log\left|\frac{\vartheta_4}{\eta^3}(4iU)\right|^2 - \log\left|\frac{\vartheta_4}{\eta^3}(2iU)\right|^2 - \pi \operatorname{Re} U \tag{C.20}$$

and the moduli dependent part of the gauge coupling threshold corrections is

$$\begin{aligned}\Lambda_{2,p} = -16\pi^2 \operatorname{Tr}(Q^2)\Bigg[(2-p)\sum_{j=1}^{3}\Bigg(&\pi \operatorname{Re} U_j \\ &+ \log\big[(\operatorname{Re} U_j)(\operatorname{Re} T_j)\mu^2\big] - \log\left|\frac{\vartheta_4}{\eta^3}(2iU_j)\right|^2\Bigg) \\ &+ q\left(\log\left|\frac{\vartheta_4}{\eta^3}(2iU_1)\right|^2 - \log\left|\frac{\vartheta_4}{\eta^3}(4iU_1)\right|^2 + \pi \operatorname{Re} U_1\right)\Bigg],\end{aligned} \tag{C.21}$$

with a β-function coefficient,

$$b_p = -3(p-2), \tag{C.22}$$

that can be easily extracted from the previous expression.

C.3. Threshold corrections in the heterotic models

We consider separately the contributions from left- and right-mover oscillators in (5.22). The left-mover contributions read

$$\begin{aligned}\Lambda_{\text{left}} = \frac{1}{8\eta^3}\sum_{i=1}^{3}\Bigg[&\left(\partial_\tau\left(\frac{\vartheta_3}{\eta}\right)\vartheta_3\vartheta_4^2 - \partial_\tau\left(\frac{\vartheta_4}{\eta}\right)\vartheta_4\vartheta_3^2\right)(-1)^{m_i+n_i}\Lambda_i\left|\frac{4\eta^2}{\vartheta_2^2}\right|^2 \\ &+ \left(\partial_\tau\left(\frac{\vartheta_3}{\eta}\right)\vartheta_3\vartheta_2^2 - \partial_\tau\left(\frac{\vartheta_2}{\eta}\right)\vartheta_2\vartheta_3^2\right)\Lambda_i^{m_i+\frac{1}{2},n_i+\frac{1}{2}}\left|\frac{4\eta^2}{\vartheta_4^2}\right|^2 \\ &- \left(\partial_\tau\left(\frac{\vartheta_2}{\eta}\right)\vartheta_2\vartheta_4^2 - \partial_\tau\left(\frac{\vartheta_4}{\eta}\right)\vartheta_4\vartheta_2^2\right)(-1)^{m_i+n_i}\Lambda_i^{m_i+\frac{1}{2},n_i+\frac{1}{2}}\left|\frac{4\eta^2}{\vartheta_3^2}\right|^2\Bigg].\end{aligned} \tag{C.23}$$

Making use of the identities (D.4)–(D.8), we get after some small algebra

$$\begin{aligned}\Lambda_{\text{left}} = \frac{\pi i}{2\bar\eta^6}\Big[&(-1)^{m_i+n_i}\hat Z_i\bar\vartheta_3^2\bar\vartheta_4^2 - \hat Z_i^{m_i+\frac{1}{2},n_i+\frac{1}{2}}\bar\vartheta_2^2\bar\vartheta_3^2 \\ &- (-1)^{m_i+n_i}\hat Z_i^{m_i+\frac{1}{2},n_i+\frac{1}{2}}\bar\vartheta_2^2\bar\vartheta_4^2\Big],\end{aligned} \tag{C.24}$$

where the toroidal lattice sums $\hat Z_i \equiv |\eta|^4\Lambda_i$ are provided by (5.24)–(5.25), after Poisson resummation in m_1 and m_2.

Regarding the contributions from the right-mover fermionic oscillators, we get

$$\Lambda_{\text{right}} = \left(Q_{SO(32)}^2 - \frac{1}{4\pi\tau_2}\right)\frac{1}{2}\sum_{a,b}\frac{\bar\vartheta[{}^a_b]^{16}}{\bar\eta^{16}}$$

$$= -\frac{1}{8\pi^2}\frac{\bar{\vartheta}[{}^a_b]''\bar{\vartheta}[{}^a_b]^{15}}{\bar{\eta}^{16}} - \frac{1}{8\pi\tau_2}\sum_{a,b}\frac{\bar{\vartheta}[{}^a_b]^{16}}{\bar{\eta}^{16}}. \tag{C.25}$$

Making use of relations (D.5)–(D.12), these terms can be rearranged in the very compact expression

$$\Lambda_{\text{right}} = \frac{\bar{E}_4(\bar{E}_4\bar{\hat{E}}_2 - \bar{E}_6)}{12\bar{\eta}^{16}}, \tag{C.26}$$

corresponding to the modular covariant derivative of $\bar{E}_8$.

Putting all together we then arrive to the final expression for the gauge kinetic threshold corrections to the $SO(32)$ heterotic model,

$$\begin{aligned}\Lambda_2 &= \frac{i}{4\pi}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\Lambda_{\text{left}}\Lambda_{\text{right}} \\ &= -\frac{1}{96}\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\sum_{i=1}^{3}\Big[(-1)^{m_i+n_i}\hat{Z}_i\bar{\vartheta}_3^2\bar{\vartheta}_4^2 - \hat{Z}_i^{m_i+\frac{1}{2},n_i+\frac{1}{2}}\bar{\vartheta}_2^2\bar{\vartheta}_3^2 \\ &\quad - (-1)^{m_i+n_i}\hat{Z}_i^{m_i+\frac{1}{2},n_i+\frac{1}{2}}\bar{\vartheta}_2^2\bar{\vartheta}_4^2\Big]\frac{\bar{E}_4(\bar{\hat{E}}_2\bar{E}_4 - \bar{E}_6)}{\bar{\eta}^{24}}.\end{aligned} \tag{C.27}$$

In the limit of large volume, $\operatorname{Re} T_i \gg 1$, or equivalently $q \to 0$ and $n_i = 0$, only degenerate orbits consisting of A matrices (5.25) with zero determinant in the sector $(h, g) = (1, 0)$ contribute to the toroidal lattice sums. Following [33], then we can pick an element A_0 in each orbit and to integrate its contribution over the image under V of the fundamental domain, for all $V \in SL(2)$ yielding $A_0 V \neq A_0$. The representatives can be chosen to be,

$$A_0 = \begin{pmatrix} 0 & j+\frac{1}{2} \\ 0 & p \end{pmatrix}, \tag{C.28}$$

enforcing the identification

$$(j, p) \sim (-j-1, -p). \tag{C.29}$$

With this representation, $A_0 V' = A_0 V''$ if and only if

$$V' = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} V''.$$

Therefore, the contributions are integrated over $\{\tau_2 > 0, |\tau_1| < \frac{1}{2}\}$, and the double covering is taking into account by summing over all p and j,

$$I_d = (\operatorname{Re} T)\int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \int_0^\infty \frac{d\tau_2}{\tau_2^2}\sum_{j,p}\exp\left(-\frac{\pi \operatorname{Re} T}{\tau_2 \operatorname{Re} U}\left|j+\frac{1}{2}+iU_i p\right|^2\right). \tag{C.30}$$

This is exactly the same expression as (C.10), so the contributions of the degenerate orbits perfectly match the perturbative type I threshold corrections,

$$I_d = \log\left|\frac{\vartheta_4}{\eta^3}(2iU_i)\right|^2 - \pi \operatorname{Re} U_i - \log\big[(\operatorname{Re} U_i)(\operatorname{Re} T_i)\big]. \tag{C.31}$$

Analogously, in the limit $q \to 0$ but $n_i \neq 0$ also the non-degenerate orbits in the sector $(h, g) = (1, 0)$ contribute. The representative in this class can be chosen to have the form

$$A_0 = \begin{pmatrix} k & j + \frac{1}{2} \\ 0 & p \end{pmatrix}, \tag{C.32}$$

with $k > j \geqslant 0$, $p \neq 0$. For these, $V' \neq V''$ implies $A_0 V' \neq A_0 V''$, and therefore these contributions must be integrated over the double cover of the upper half plane ($\tau_2 > 0$),

$$\begin{aligned} I_{nd} = 2(\operatorname{Re} T) \sum_{0 \leqslant j < k, p \neq 0} e^{2\pi T k p} \int_{-\infty}^{\infty} d\tau_1 \int_0^{\infty} \frac{d\tau_2}{\tau_2^2} (-1)^k \\ \times \exp\left[-\frac{\pi \operatorname{Re} T}{\tau_2 \operatorname{Re} U} \left| k\tau + j + \frac{1}{2} + ipU \right|^2 \right]. \end{aligned} \tag{C.33}$$

Evaluating the Gaussian integral over τ_1 and summing on j, one gets

$$\begin{aligned} I_{nd} = 2 \sum_{0 < k, p \neq 0} e^{2\pi T k p} \int_0^{\infty} d\tau_2 \sqrt{\frac{(\operatorname{Re} U)(\operatorname{Re} T)}{\tau_2^3}} (-1)^k \\ \times \exp\left[-\frac{\pi \operatorname{Re} T}{\tau_2 \operatorname{Re} U} (k\tau_2 + p \operatorname{Re} U)^2 \right], \end{aligned} \tag{C.34}$$

and the contribution of this sector becomes

$$I_{nd} = \log \left| \frac{\vartheta_4}{\eta^3} (2iT) \right|^2 - \pi \operatorname{Re} T. \tag{C.35}$$

It corresponds to E1 instanton corrections in the type I $SO(32)$ dual model. Indeed, expanding the η-function in (C.35), I_{nd} can be expressed as

$$I_{nd} = -2 \sum_{n=1}^{\infty} (-1)^n \log\left(1 - e^{-2\pi n T}\right) + \text{c.c.}, \tag{C.36}$$

which should correspond to a sum over the contributions of E1-instantons wrapping n times the $(1, 1)$-cycle associated to T, a fact that would be very interesting to verify explicitly. Notice that the dependence on T perfectly agrees with the general arguments in [16] for the mirror type IIA picture.

Appendix D. Some useful formulae

Poisson summation formula:

$$\sum \operatorname{Exp}\left[-\pi (m-b)^T A (m-b)\right] = \frac{1}{\sqrt{\det A}} \sum \operatorname{Exp}\left[-\pi n^T A^{-1} n + 2i\pi b^T n\right]. \tag{D.1}$$

Modular identities:

$$\vartheta_3'' \vartheta_3^3 - \vartheta_4'' \vartheta_4^3 - \vartheta_2'' \vartheta_2^3 = 0, \tag{D.2}$$

$$\vartheta_3'' \vartheta_3 \vartheta_4^2 - \vartheta_4'' \vartheta_4 \vartheta_3^2 = -4\pi^2 \eta^6 \vartheta_2^2, \tag{D.3}$$

$$\vartheta_2 \vartheta_3 \vartheta_4 = 2\eta^3, \tag{D.4}$$

$$\vartheta_2'' = 4\pi i \partial_\tau \vartheta_2 = -\frac{\pi^2}{3}\vartheta_2\big(E_2 + \vartheta_3^4 + \vartheta_4^4\big), \tag{D.5}$$

$$\vartheta_3'' = 4\pi i \partial_\tau \vartheta_3 = -\frac{\pi^2}{3}\vartheta_3\big(E_2 + \vartheta_2^4 - \vartheta_4^4\big), \tag{D.6}$$

$$\vartheta_4'' = 4\pi i \partial_\tau \vartheta_4 = -\frac{\pi^2}{3}\vartheta_4\big(E_2 - \vartheta_2^4 - \vartheta_3^4\big). \tag{D.7}$$

Eisenstein series:

$$E_2 = \hat{E}_2 + \frac{3}{\pi\tau_2} = \frac{12}{i\pi}\partial_\tau \log\eta = 1 - 24q - \cdots, \tag{D.8}$$

$$E_4 = \frac{1}{2}\big(\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8\big) = 1 + 240q + \cdots, \tag{D.9}$$

$$E_6 = \frac{1}{2}\big(\vartheta_2^4 + \vartheta_3^4\big)\big(\vartheta_3^4 + \vartheta_4^4\big)\big(\vartheta_4^4 - \vartheta_2^4\big) = 1 - 540q - \cdots, \tag{D.10}$$

$$E_8 = E_4^2 = \frac{1}{2}\big(\vartheta_2^{16} + \vartheta_4^{16} + \vartheta_3^{16}\big) = 1 + 480q + \cdots, \tag{D.11}$$

$$E_{10} = E_4E_6 = -\frac{1}{2}\big[\vartheta_2^{16}\big(\vartheta_3^4 + \vartheta_4^4\big) + \vartheta_3^{16}\big(\vartheta_2^4 - \vartheta_4^4\big) - \vartheta_4^{16}\big(\vartheta_2^4 + \vartheta_3^4\big)\big]. \tag{D.12}$$

Series expansions:

$$\log(1-Q) = -\sum_{n=1}\frac{Q^n}{n}, \tag{D.13}$$

$$\log(1+Q) = -\sum_{n=1}(-1)^n\frac{Q^n}{n}, \tag{D.14}$$

$$\log\vartheta_2 = \log 2q^{1/8} + \sum_{n=1}\log\big(1-q^n\big) + 2\sum_{n=1}\log\big(1+q^n\big), \tag{D.15}$$

$$\log\vartheta_4 = \sum_{n=1}\log\big(1-q^n\big) + 2\sum_{n=1}\log\big(1-q^{n-\frac{1}{2}}\big), \tag{D.16}$$

$$\log\eta = \log q^{1/24} + \sum_{n=1}\log\big(1-q^n\big), \tag{D.17}$$

$$\frac{1+Q}{1-Q} = 1 + \sum_{m=1}Q^m, \tag{D.18}$$

$$\frac{Q-1}{Q+1} = -1 - 2\sum_{m=1}(-1)^m Q^m. \tag{D.19}$$

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